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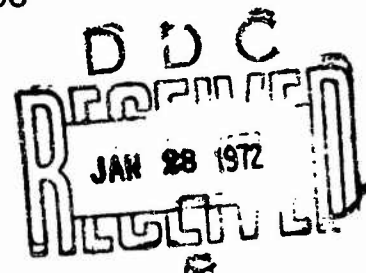
# INTERACTIVE PROGRAM FOR ANALYSIS AND DESIGN PROBLEMS IN ADVANCED COMPOSITES TECHNOLOGY

T. A. CRUSE  
J. L. SWEDLOW

TECHNICAL REPORT AFML-TR-71-268

DECEMBER 1971

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REV.	REAL. NO./S. SERIAL
A	

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## DOCUMENT CONTROL DATA - R &amp; D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Carnegie-Mellon University Mechanical Engineering Department Pittsburgh, Pennsylvania 15213		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP N/A	
3. REPORT TITLE Interactive Program for Analysis and Design Problems in Advanced Composites Technology			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Final Report (November 1969 - October 1971)			
5. AUTHOR(S) (First name, middle initial, last name) T. A. Cruse and J. L. Swedlow			
6. REPORT DATE November 1971	7a. TOTAL NO. OF PAGES 296	7b. NO. OF REFS 76	
8a. CONTRACT OR GRANT NO. F33615-70-C-1146	8b. ORIGINATOR'S REPORT NUMBER(S) AFML-TR-71-268		
8c. PROJECT NO. 6169 CW	8d. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)		
10. DISTRIBUTION STATEMENT Approved for public release: Distribution unlimited			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY Advanced Composites Division (AFML/LC) Air Force Materials Laboratory Wright-Patterson Air Force Base, Ohio 45433	
13. ABSTRACT The Carnegie-Mellon University team has completed the initial Interactive Program in Advanced Composites Technology. The program has had significant impact as the CMU team, working closely with engineers from industry, has made significant technical progress in several problem areas of current importance. Results on these problems are reported in this Report. During the past year an experimental program in the fracture of advanced fiber composites has been completed. The experimental program has given direction to additional experimental and theoretical work. A synthesis program for designing low weight multifastener joints in composites is proposed, based on extensive analytical background. A number of failed joints have been thoroughly analyzed to evaluate the failure hypothesis used in the synthesis procedure. Finally, the Report includes new solution methods for isotropic and anisotropic (mid-plane symmetric) laminates using the boundary-integral method. The solution method offers significant savings of computer core and time for important problems.			

DD FORM 1473  
1 NOV 65

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Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Composite Materials						
Education						
Mechanically Fastened Joints						
Fracture Mechanics						
Stress Concentrations						
Numerical Solution Methods						
Anisotropic Elasticity						
Optimization						
Synthesis						

UNCLASSIFIED

Security Classification



AFML-TR-71-268

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this document may be better  
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DECEMBER 1971

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## FOREWORD

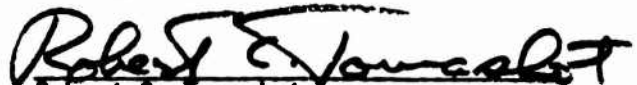
This report describes work performed in the Department of Mechanical Engineering, Carnegie-Mellon University, Pittsburgh, Pennsylvania, 15213, under Air Force Contract F33615-70-C-1146, Project 6169 CW; Subject: "Research for the development of design and analytical techniques for advanced composite structures". This work was accomplished between 1 November 1969 and 31 October 1971. The Program Manager for the Air Force Materials Laboratory was Mr. George E. Husman, AFML/MAC. The research projects in this report were undertaken by several graduate students under the direction of the Principal Investigators, Drs. T. A. Cruse and J. L. Swedlow and it is a pleasure to acknowledge Messrs. H. J. Konish, Jr. (fracture project), J. P. Waszco (mechanically fastened joints), W. B. Bamford (numerical studies using integral equations) and S. J. Marulis (optimization); it is also a pleasure to acknowledge the work on advanced topics in integral equations performed on a consulting basis by Professor Frank J. Rizzo, University of Kentucky.

Since the emphasis of the program was on interaction with industry it is a pleasure to acknowledge Messrs. C. W. Rogers, P. D. Shockey, M. E. Waddoups, R. B. Pipes and many others in the advanced composites group of Convair Aerospace Division, General Dynamics, Fort Worth, Texas; also the staffs at Boeing/Vertol, Grumman Aerospace, North American Rockwell and Southwest Research Institute.

Portions of the reported work were also supported by NASA Grant NGR-39-002-023 (fracture project) and by General Dynamics P. O. No. 499527 (joint design). The authors would finally like to acknowledge Miss Kathleen Sokol for her careful and selfless devotion to the preparation of this document.

This report was submitted by the Authors on 6 December 1971.

This technical report has been reviewed and is approved,



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Technical Area Manager  
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## ABSTRACT

The Carnegie-Mellon University team has completed the initial Interactive Program in Advanced Composites Technology. The program has had significant impact as the CMU team, working closely with engineers from industry, has made significant technical progress in several problem areas of current importance. Results on these problems are reported in this Report. During the past year an experimental program in the fracture of advanced fiber composites has been completed. The experimental program has given direction to additional experimental and theoretical work. A synthesis program for designing low weight multifastener joints in composites is proposed, based on extensive analytical background. A number of failed joints have been thoroughly analyzed to evaluate the failure hypothesis used in the synthesis procedure. Finally, the Report includes new solution methods for isotropic and anisotropic (mid-plane symmetric) laminates using the boundary-integral method. The solution method offers significant savings of computer core and time for important problems.

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# LIST OF SYMBOLS AND NOTATION

CHAPTER	SECTION	SYMBOL	DESCRIPTION
II	1	$r, \theta$	Polar coordinates, with the origin at the crack-tip. (See Fig. 1).
		$x, y$	Cartesian coordinates, with the origin at the crack-tip; these are the global coordinates. (See Fig. 1).
		$1, 2$	Cartesian coordinates based on the material principal directions (See Fig. 1).
		$B$	Thickness of the three-point bend specimen.
		$S$	Span of the three-point bend specimen.
		$W$	Depth of the three-point bend specimen.
		$a$	Crack-length.
		$\alpha$	Angle of rotation between the global and lamina coordinate systems (See Fig. 1).
		$\sigma_{x,y}$	Tensile stresses in the global coordinate system.
		$v$	Displacement normal to the crack-axis.
		$P$	Applied load on the three-point bend specimen.
		$\mu_i$	Roots of the characteristic equation.
		$a_{ij}$	Components of the global compliance matrix.

CHAPTER	SECTION	SYMBOL	DESCRIPTION
II	1	$K_{I,II}$	Stress intensity factors corresponding to symmetric and anti-symmetric loading respectively. A subscript c indicates a critical value of K, i.e., a value at which the crack propagates catastrophically.
	2	$K_Q$	A candidate value of the critical stress-intensity factor.
		$\bar{K}_Q$	The average value of $K_Q$ for a given laminate, obtained by averaging the $K_Q$ values obtained for several specimens of the laminate.
		$G_Q$	Strain-energy release rate.
		$M_0$	Initial slope of the experimental plot of load vs. specimen deflection (See Fig. 12).
		$P_S$	Load corresponding to the intersection of the secant of slope $M_0$ with the curve of load vs. specimen deflection (See Fig. 12).
		$P_Q$	Applied load at which crack propagation occurred.
		$a$	Crack-length (See Fig. 1).
		$\alpha$	Angle of rotation between the global and lamina coordinate systems (See Fig. 1).

CHAPTER	SECTION	SYMBOL	DESCRIPTION
III	1	e	Specimen edge distance.
		d	Bolt bearing diameter.
		s	Total specimen width.
		ℓ	Total specimen length.
		t	Specimen thickness
		$\sigma_i$	$i^{\text{th}}$ principal lamina stress.
		$\sigma_{iu}$	$i^{\text{th}}$ principal ultimate lamina stress.
		DIST	Normalized distortional energy.
	2	$F_{TU}$	Effective tension strength.
		$F_{SU}$	Effective shear-out strength.
		$F_{BRU}$	Effective bearing strength.
		S	Bolt bearing specimen width.
		E	Bolt bearing specimen edge distance.
		t	Bolt bearing specimen thickness.
		D	Bolt bearing specimen diameter.
		L'	Bolt bearing specimen length.
		L	Total joint length.
		N	Number of bolts per column.
		F	Maximum load to be carried per column of bolts



CHAPTER	SECTION	SYMBOL	DESCRIPTION
III	2	Subscripts and Superscripts	
		m	Main plate
		s	Splice plate
		B	Bolt material
		u	Ultimate allowable
		t	Tension
		c	Compression

CHAPTER	SECTION	SYMBOL	DESCRIPTION
IV	2	$W(X_i)$	Weight function of a composite plate.
		$X_i$	Variables of a composite plate.
		$F_j(X_i)$	Constraint functions on $W(X_i)$ .
		$\lambda_j$	Lagrange multipliers for the constraint functions.
		P	Objective function for minimization.
	3	M	Applied torsional moment.
		a, b	Semi-major and semi-minor axes of the ellipse.
		$\tau$	Shear stress.
	4	t	Thickness of the bolt bearing specimen.
		S	Width of the bolt bearing specimen.
		E	Distance from the center of the bolt hole to the near edge of the bolt bearing specimen.
		D	Diameter of the bolt hole in the bolt bearing specimen.
		XL	Distance from the bolt hole to the far edge of the bolt bearing specimen.
		P	Load applied to the bolt bearing specimen.
		$F^{tu}$	Failure stress for the tension mode failure of the bolt bearing specimen.
		$F^{su}$	Failure stress for the shear out mode failure of the bolt bearing specimen.

CHAPTER	SECTION	SYMBOL	DESCRIPTION
IV	4	$p^{bu}$	Failure stress for the bearing mode failure of the bolt bearing specimen.
		L	Percentage of 0° plies in the bolt bearing specimen.
		M	Percentage of 90° plies in the bolt bearing specimen.
		N	Percentage of $\pm 45^\circ$ plies in the bolt bearing specimen.

CHAPTER	SECTION	SYMBOL	DESCRIPTION
V	2		
		$\sigma_x, \sigma_y, \sigma_{xy}$	Stress field.
		$\epsilon_x, \epsilon_y, \gamma_{xy}$	Strain field.
		$u_x, u_y$	Displacement field.
		$\beta_{ij}$	Material compliances.
		$E_x, E_y, G_{xy}$	Axial, shear moduli.
		$\nu_{xy}, \eta_{xy,y}, \eta_{xy,x}$	Coupling coefficients.
		$\alpha_{ij}$	Material stiffness.
		$F, F_1, F_2$	Stress functions.
		$z, z_k$	Characteristic directions.
		$\mu$	Roots of the characteristic equation.
		$i$	$\sqrt{-1}$ .
		$\phi_1, \phi_2$	Derivatives of $F_1(z_1), F_2(z_2)$ .
		$R[ ]$	Real part of $[ ]$ .
		$p_k, q_k$	Constants.
		$t_x, t_y$	Traction components.
		$n_x, n_y$	Outward normal.
		$\phi_{ik}$	Stress function.
		$\delta_{ij}$	Kronecker delta.
		$A_{jk}, C_{jk}, D_{jk}$	Complex, real constants.
		$U_{ji}$	Fundamental displacement tensor.
		$T_{ji}$	Fundamental traction tensor.
		$P_{ik}, Q_{ik}$	Complex constants.
		$\epsilon_{ij}$	Strain field.
		$S_{j\ell i}, D_{j\ell i}$	Tensor kernels for $\epsilon_{ij}$ .

CHAPTER	SECTION	SYMBOL	DESCRIPTION
V	1	$x_i$	Cartesian coordinates.
		$U_{ij}$	Singular influence tensor.
		$P(x), Q(x)$	Boundary points.
		$r(P,Q)$	Distance between $P(x), Q(x)$ .
		$\nu$	Poisson's ratio.
		$\mu$	Shear modulus.
		$\pi$	Pt.
		$\delta_{ij}$	Kronecker delta.
		$u_i$	Displacement vector.
		$t_i$	Traction vector.
		$\sigma_{ij}$	Stress tensor.
		$n_j$	Unit outward normal vector.
		$T_{ij}$	Singular influences tensor.
		$\partial R$	Surface of the body, $R$ .
		$N$	Number of boundary segments.
		$P_m, Q_n$	Discrete boundary points.
		$[I]$	Identity matrix.
		$[\Delta T], [\Delta U]$	Coefficient matrices.
		$\{t\}$	Traction vector.
		$\{u\}$	Displacement vector.
		$\Delta S_{kij}, \Delta D_{kij}$	Integrals of influence tensors.

CHAPTER	SECTION	SYMBOL	DESCRIPTION
V	3	$\epsilon_x$	x-direction strain.
		$U_x(1),$ $U_x(2)$	x-direction displacement at segment number 1; segment number 2.
		$l$	Distance between midpoints of adjacent boundary segments, as shown in Fig. 1 and Fig. 2.
		$Y^*$	y-coordinate of last valid data point obtained for in- terior solution points, be- fore data diverge from the theoretical solution.
		SCF	Stress concentration factor.
		$a$	Semi-major axis of an ellipse.
		$b$	Semi-minor axis of an ellipse.
		$c$	Semi-focal distance.
		$\sigma_x$	x-direction stress.
		$\sigma_y$	y-direction stress.
	4	$C_{ijkl}$	Elastic constant tensor.
		$U_k$	Displacement vector.
		$\tau_{ij}$	Stress tensor.
		$e_{ij}$	Strain tensor.
		$L_{ij}$	Second order, linear operator.
		$U_{ij}$	Singular influence tensor.
		$\underline{x}, \underline{y}$	Spatial points.
		$\partial R, \Gamma$	Surfaces.
		$T_{ij}$	Singular influence tensor
		$t_i$	Traction vector.

CHAPTER	SECTION	SYMBOL	DESCRIPTION
V	4	$\epsilon$	Radius.
		$n_k$	Unit outward normal vector.
		$\pi$	Pi.
		$\Delta y$	Laplacian at $y$ .
		$\xi$	Vector.
		$\Omega_\xi$	Sphere of unit radius $\xi = 0$ .
		$p^{kj}(\xi)$	Inverse of $Q_{ik}$ .
		$Q_{ik}(\xi)$	Characteristic form of $L_{ik}$ .
		$R$	Vector, $x - y$ .
		$\psi$	Angle between $R$ and $\xi$ .
		$A_{jk}$	Tensor.
		$\epsilon_{ijk}$	Alternating symbol.
		$\text{Det } Q$	Determinant of $Q_{ij}$ .
		$\lambda, \mu$	Lame constants.
		$\alpha, \beta$	Material constants.
		$\psi_i$	Angle between $R$ and $x_i$ ; polynomials in $\xi, n$ .
		$c$	Constant.
		$U_1, U_2, U_3$	Functions of $y, z$ .
		$S_x, S_y, S_z$	Surfaces with normals in $x, y, z$ directions.
		$\ell$	Length of specimens.
		$\hat{U}_{ij}, \hat{T}_{ij}$	Influence tensors, independent of $x$ .
		$f_j$	Vector function.
		$w$	Lamina width.
		$t$	Lamina thickness.

## CHAPTER I

### SUMMARY OF THE INTERACTIVE PROGRAM

#### 1.1 INTRODUCTION

The Carnegie-Mellon University team of faculty and students has developed a unique program of interaction between the University team, the Air Force Materials Laboratory, and certain aerospace industries, notably General Dynamics, Convair Aerospace Division (Fort Worth). The interactive program has focused on the application of mechanics capabilities of the CMU team to the stress and strength analysis of advanced fiber composite structures. The broad objectives of the program are the following:

1. Creation of new and effective means of communication and interaction between CMU and General Dynamics and other aerospace industries.
2. Involvement of the CMU team in the solution of fundamental engineering problems arising from the application of advanced composites in aerospace structures.
3. Development by the CMU team of new stress analysis capabilities and results, strength criteria, design information and educational material for advanced composites technology.

To accomplish these goals, a two year effort was initiated at CMU under Air Force sponsorship in November, 1969. The two year program has been completed and has successfully met the goals delineated above. The purpose of this Final Report is to summarize the achievements of the Interactive Program. This first Chapter discusses results for all of the



objectives. Following Chapters discuss in detail the results for objectives 2 and 3.

The principal investigators for this program originally adopted the position that the second objective would be promoted through extensive contacts with industry, and that student members of the CMU team would be select senior undergraduate and first- and second-year graduate students. This position precluded supporting Ph.D. and faculty research by the program. However, two student members of the CMU team have passed the Ph.D. qualifying exam and are doing their research based on their project experience (Fracture of Composites; Design of Mechanically Fastened Joints). To date, five undergraduate and fifteen graduate students have participated to some extent in the Interactive Program. Faculty other than the Principal Investigators have participated in the educational program to become familiarized with advanced composites technology and to lend particular expertise as needed.

## 1.2 FIRST AND SECOND YEAR PROGRAMS

### 1.2.1 Phase I

During each year the Interactive program has been divided into three phases: education, project research, and reporting. The education phase is based on a Fall Semester course, *Mechanics of Fiber Composite Materials*. The purpose of the course is to bring the students "up-to-speed" in advanced composites technology such that they can contribute significantly to the solution of engineering problems. In the second year of the program, Dr. Cruse offered an advanced course, *Two Dimensional Anisotropic Elasticity*, which was based on the analytical solution of membrane problems of composites using the complex variable approach. A

summary of the educational program is included in Appendix I, Chapter I. This summary includes course outlines and descriptions, references, and homework problem titles.

The emphasis in the course work is on the identification of state-of-the-art knowledge and on solving meaningful homework problems. An example of this is the use of the "pressure vessel" problem. Students are asked to find the optimal winding angle ( $\pm\alpha$ ) and maximum pressure for a cylindrical pressure vessel, using a fixed material (e.g. graphite-epoxy) and each of the proposed failure criteria. The problem forces the student to exercise lamination theory and allows a comparison of the allowable pressures.

Another important problem area that was used is the stress concentration factors in composite plates subject to in-plane loading. The fact that these factors are always higher than for isotropic materials is emphasized. The discussion leads to other measures of strength such as associated with sharp flaws.

The students make considerable use of the computer and in-house analysis programs such as finite element and boundary-integral methods for boundary value problems and a pattern search program for optimization and synthesis. Through all of the exercises the student develops insight into the fundamental mechanics questions and spends very little time on the nature of the analysis programs.

### *1.2.2 Phase II*

The second phase lasts through the Spring Semester and sometimes, for significant problems, through the summer. The purpose of the second phase is to involve the students in engineering problems in advanced

composites technology. The students, with faculty and industry guidance, select problems of interest to the student and industry. The process of problem selection for new members of the team was a major portion of the second half of the Fall Semester course.

In January of each year Dr. Cruse presented the project problems at General Dynamics for evaluation and recommendations. At the same time engineers at General Dynamics were identified who would act as the industrial contact for the student working on a particular project.

A major portion of the budget of the Program was devoted to travel support. The reasoning is that the CMU team, to be effective, must have considerable visibility of the industrial problems in composites technology. Thus, during the second phase the University team made several visits to industrial locations, technical meetings, program reviews, and special Air Force programs. These trips have also served to give the CMU team visibility as a group doing significant work in the area of advanced composites technology. A complete list of trip report titles is presented in Appendix II, Chapter I. This list illustrates the breadth and depth of the CMU team contacts with other teams in the technology area.

In the first year of the program, the CMU team made a group visit to General Dynamics. This trip was for presenting student progress reports on their projects; it also was a chance for the student members of the team to see manufacturing and test programs in progress. In the second year, the new member of the CMU team to pursue project work visited Boeing/Vertol to see their advanced composites manufacturing and test program. However, the rest of the members of the team doing project research were in their second year, and thus a team visit to industry

was not made during the second phase of the second year.

The telephone is used heavily during the second phase. By identifying engineers, perhaps at different industrial locations, who had an interest in the student project problem, each student could ask questions and receive advice, data, and evaluation without the necessity of a full visit with the engineer. The CMU team found that continual contact with engineers played a major role in the success of the Interactive Program.

### *1.2.3 Phase III*

The third phase of the program is the reporting phase for each project problem. Each student, upon reaching a major milestone, or when completing his participation in the program, is required to provide a written project report which is typed and filed. Thus, the reporting phase is interweaved throughout the program. A list of the titles of all reports generated and on file is given in Appendix III, Chapter I. The reports contain major homework problem solutions from Phase I work, project proposals and progress reports, tutorial material, and final project reports.

Some of the project reports are significant enough to be published in technical journals [1,2]<sup>1</sup> and to be presented at technical meetings [3,4]. In addition other reports have been submitted for presentation to the 13th AIAA/ASME Structures, Structural Dynamics and Materials Conference [5,6], while another has been accepted for the 1972 ASTM

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<sup>1</sup>References are denoted by brackets [ ] and are found at the end of each major segment of this Report.

meeting on composites [7]. These papers serve to give to CMU team greater visibility in the composites community as well as to report important results.

In addition to the above major reports, the program had a requirement for monthly letter reports, at the request of the Principal Investigators. These reports required monthly student progress reports while the students were doing project research. The monthly report served to force each member of the team to be fully aware of his own and others' progress. In addition the reports kept the industrial team informed of project progress.

At the end of the summer, each year, the CMU team prepared final project reports which were presented at the Air Force Materials Laboratory and at General Dynamics. This final reporting has been the most important facet of Phase III as the CMU team seeks critical review of its programs by the active researchers and engineers at both locations. The final report meetings served as the focal point for examination of progress, but they also provided an opportunity to explore new areas of project work, team emphasis, and industrial support.

### *1.3 RESEARCH PROJECTS COMPLETED*

A sizeable number of project research problems have been solved to date and the titles are listed in Appendix II, Chapter I. Listed below are the major project titles, the responsible investigator, a summary of the project and project reports as found in the SM file in the Mechanical Engineering Department. The following Chapters of this Final Report present in detail the major accomplishments of each project.

#### *1.3.1 Fracture of Advanced Composites* (H. J. Konish, Jr.)

This project includes analytical and experimental investigations of the fracture of moderately thick graphite/epoxy specimens. Information to date has been very encouraging in that a considerable amount of linear elastic fracture mechanics theory seems applicable to the material. (SM Reports 31, 41, 53, 64, 74, 80, 81; work in progress).

#### *1.3.2 Strength of Mechanically-Fastened Joints* (J. P. Waszczak)

This project has gone from the analysis of single-fastener test coupons to the analysis of joints with many fasteners. Due to the weight penalty associated with these joints, a program has been begun to develop a synthesis procedure for designing multifastener joints. This program has a strong coupling with the engineering team at General Dynamics. (SM Reports 28, 34, 63, 76; work in progress).

#### *1.3.3 Optimization Methods* (S. J. Marulis; Ford Motor Co.)

The project was to investigate the use of an in-house, pattern-search optimization method for composite design problems. The design of mechanically-fastened joints was considered, using the in-house program. An effort to couple the optimization program to the available finite element program was unsuccessful but may be completed in the future. The optimization program has been found suitable, if not optimal, for use by Mr. Waszczak in his project research. (Report SM-71; work suspended).

#### *1.3.4 Boundary-Integral Equation Solution Methods* (T. A. Cruse, W. H. Bamford, F. J. Rizzo)

Three separate efforts have been completed in this area. The first reported is the development of an isotropic, two dimensional boundary-integral equation method and a subsequent investigation of its

ability to model cutouts under tension. The second reported is the development of a boundary-integral method for fully-anisotropic (mid-plane symmetric) laminates. (Currently, the anisotropic program is being verified on cutout problems and some of these results are reported.) The third, completed by Prof. F. J. Rizzo of the University of Kentucky, concerns solutions to Kelvin's problem in anisotropic three dimensional bodies, and the interlaminar shear problem. (SM Reports 45, 50, 66, 68, 70, 72; work in progress).

#### *1.4 EVALUATION AND RECOMMENDATIONS*

It is clear that the goals of the Interactive Program at CMU have been met. The project reports contained in this Final Report give ample evidence of the extent to which the CMU team has become competent in research and application problems in advanced composites technology. There now exists considerable interaction and support between the General Dynamics team and the CMU team. In particular, General Dynamics has provided test specimens for the Fracture Program and a summer contract for the Joint Project.

However, the level of confidence in the CMU team expressed by General Dynamics has come late in the program. Communication and interaction took place during the first year of the program but the depth of both was not satisfying to either team. One reason for this was that during the first year the CMU team was just coming up to speed in advanced composites technology. However, based on the results of the program review at the end of the first year, the support from the General Dynamics team increased rapidly. The other reason for the slow start was the lack

of *constant* contact between the CMU team and the General Dynamics team. During the second year, much more contact was made, principally by Dr. Cruse visiting General Dynamics and liberal use of the telephone. Frequent personal contacts are critically important to the success of an interactive program such as ours.

The impact to date on the educational program at CMU has been minor. The two courses cited in Appendix I plus project work (counts as course work) are the extent of highly visible composites activities in the educational program. However, seminars given by General Dynamics and AFML personnel, and by the Principal Investigators have served to make other faculty aware of the questions of materials selection, and composites in particular. During one semester Dr. Cruse taught a section of Senior Design which was concerned with the rationale for materials selection. At the present time Dr. Cruse is involved in an effort to expand the CMU Post-College Professional Education Program. This effort includes a course on fiber composites.

At a harder level to document, instructors in the basic solid mechanics courses in the Mechanical Engineering Department have the specimens and knowledge to demonstrate simple anisotropic effects. It is hoped that more of this information can be meaningfully involved in the undergraduate courses. One of the biggest problems which mitigates against new courses in the undergraduate or graduate program is the financial state of the University. The process of cutting-back is underway and will likely last a few more years.

Finally, the question arises as to the impact the Program has had in developing graduates with a competence in advanced composites,



who will use this competence in the aerospace industry. To date this impact has been nearly zero, as most of the students who have done significant project work have yet to graduate. An early graduate with contact with the Interactive Program went to Pratt and Whitney; another graduate went to Ford Motor Company. Several graduate students with other research areas have taken one or both of the courses offered to date. Those in the Program who are still doing project work are commissioned officers in the United States Army. Thus the personnel impact will require more time to develop.

Two years ago, CMU had no active research in the area of advanced composites. In that period the CMU team has developed an effective education - project program that is closely related to fundamental engineering problems in advanced composites technology. Members of the CMU team have presented and published an increasing number of research papers, and have participated in several Air Force review meetings. The depth and breadth of research accomplishments are reported in the remaining Chapters of this Final Report. Other measures of the Program require additional time to mature.

## 1.5 REFERENCES

- [1] J. P. Waszczak, T. A. Cruse, "Failure Mode and Strength Predictions of Anisotropic Bolt Bearing Specimens", *J. Comp. Materials* 5, (July 1971).
- [2] H. J. Konish, Jr., J. L. Swedlow, T. A. Cruse, "Experimental Investigation of Fracture in an Advanced Fiber Composite", Report SM-74, *J. Comp. Materials* (to appear).
- [3] J. P. Waszczak, T. A. Cruse, "Failure Mode and Strength Predictions of Anisotropic Bolt Bearing Specimens", Fifth St. Louis Symposium on Composite Materials (April 1971).
- [4] J. P. Waszczak, T. A. Cruse, "Failure Mode and Strength Predictions of Anisotropic Bolt Bearing Specimens", *Proceedings of the 12th AIAA/ASME Structures, Structural Dynamics and Materials Conference* (April 1971).
- [5] H. J. Konish, Jr., J. L. Swedlow, T. A. Cruse, "On Fracture in Advanced Fiber Composites", Report SM-80, (Submitted to AIAA/ASME) (October 1971).
- [6] T. A. Cruse, "Boundary-Integral Equation Solution Method for Plane Stress Analysis of Composites (Extended Abstract), Report SM-75, (Submitted to AIAA/ASME 13th Structures and Structural Dynamics Meeting) (September 1971).
- [7] H. J. Konish, Jr., J. L. Swedlow, T. A. Cruse, "A Proposed Method for Estimating Critical Stress Intensity Factors for Cross-Plyed, Mid-Plane Symmetric Composite Laminates", Report SM-81, (Extended abstract submitted to ASTM) (October 1971).

## 1.6 APPENDIX I: SUMMARY OF EDUCATIONAL MATERIAL

### I. COURSE: Mechanics of Fiber Composite Materials (*First Semester*)

- A. Course description
- B. References
- C. Course outline

### II. Project-type Homework Problems

- A. Develop computer program for calculating  $[A]$  matrix
- B. Develop computer program to reduce laminate strains to lamina stresses and strains
- C. Analyze dependence of the  $[A]$  mat 'x' terms on the fiber orientation
- D. Determine the effect of transverse tension on the inter-laminar shear stress
- E. Determine the optimum winding angle ( $\pm$ ) for a pressure vessel
- F. Evaluate the deformation in a helically-wound (+) cylinder
- G. Evaluate the finite element solution for a circular cutout
- H. Evaluate the finite element solution for a composite beam

### III. Finite Element Summary

- A. Course notes from a short course for users
- B. Usage guide for in-house finite element computer programs

### IV. COURSE: Two Dimensional Anisotropic Elasticity (*Second Semester*)

- A. Course description
- B. Some selected prepared course notes

### V. Project-type Homework Problems

- A. Isotropic
  - 1. General solutions for ring-shaped region
  - 2. Bolt-bearing solution
  - 3. Concentrated force in an infinite plate
- B. Anisotropic
  - 1. Stress concentration at an ellipse
  - 2. Hoop stress distribution at a circle
  - 3. Torsion of a prismatic member
  - 4. Point load in an infinite plate
  - 5. Bolt-bearing solution
  - 6. Stress analysis of a cracked, infinite plate

## MECHANICS OF FIBER COMPOSITE MATERIALS

### Text Material:

T. A. Cruse, *Mechanics of Laminated Fiber Composites*  
(notes in preparation)

J. E. Ashton et al, *Primer on Composite Materials: Analysis*  
Technomic (1969)

### Course Abstract:

This course deals with the stress and strength analysis of two dimensional anisotropic fiber composite structural materials. These materials have applications in structural reinforcements, pressure vessels, and aerospace structures. Typical materials that can be considered include reinforced concrete, fiberglass, and some of the new, advanced fiber composites such as boron-epoxy and graphite-epoxy. Major topics include the development of the anisotropic stiffness matrix for in-plane and out-of-plane loading of plates and shells, theories of strength and experimental procedures, and stress and displacement analysis of simple plate and shell structures. Students will participate in a number of project problems designed to involve the student in some of the real design problems associated with composite materials. Existing solution techniques such as finite elements, integral equations, and optimization computer programs, as well as analytic solution capabilities will be exercised as appropriate. The student is assumed to have completed the normal undergraduate courses in strength of materials including some introduction to the theory of elasticity.

## MECHANICS OF FIBER COMPOSITE MATERIALS

### Supplementary Reference Material:

#### BOOKS:

S. A. Ambartsumyan, *Theory of Anisotropic Plates*, Technomic (1970)

J. E. Ashton, J. M. Whitney, *Theory of Laminated Plates*, Technomic (1970)

G. S. G. Beveridge, R. S. Schechter, *Optimization: Theory and Practice*, McGraw-Hill (1970)

S. W. Tsai, et al (Editors), *Composite Materials Workshop*, Technomic (1968)

L. J. Broutman, R. H. Krock (Editors), *Modern Composite Materials*, Addison Wesley (1967)

\_\_\_\_\_, *Metal Matrix Composites*, ASTM STP 438 (1968)

\_\_\_\_\_, *Interfaces in Composites*, ASTM STP 452 (1969)

\_\_\_\_\_, *Composite Materials: Testing and Design*, ASTM STP 460 (1969)

#### REPORTS:

T. A. Cruse, J. L. Swedlow, *Interactive Program in Advanced Composites Technology: First Annual Report*, Report SM-46, Carnegie-Mellon University, Pittsburgh, Pennsylvania (1970)

M. S. Howeth, *Design, Materials and Structures*, Report SMD-028, General Dynamics, Fort Worth, Texas (1969).

S. W. Tsai, *Mechanics of Composite Materials*, AFML-TR-66-199

## MECHANICS OF FIBER COMPOSITE MATERIALS

### COURSE OUTLINE:

- I. Review of Two Dimensional elasticity (6 hours)
  - A. Stress tensor
  - B. Equilibrium
  - C. Strain Tensor
  - D. Compatibility
- II. Linear, anisotropic elasticity (5 hours)
  - A. Existence of the strain energy density
  - B. Fourth order compliance, stiffness tensors
  - C. Transformation equations
    1. Specially orthotropic
    2. Transversely isotropic
    3. Isotropic
  - D. Plane stress results
- III. Mechanics of a continuous fiber lamina (4 hours)
  - A. Manufacturing of fibers, laminae
  - B. Rules of mixtures
  - C. Summary of micromechanics results
  - D. Lamina mechanical properties
- IV. Mechanics of Laminates (12 hours)
  - A. Manufacturing of laminates
  - B. Stiffness, compliance matrices;  $[A]$ ,  $[B]$ , and  $[D]$
  - C. Strength theories
    1. Static theories: Maximum stress, strain; Distortional energy
    2. Energy tensor
    3. Fatigue
    4. Fracture
- V. Structural applications and projects (12 hours)
  - A. Finite element solution method
  - B. Joints and cutouts
  - C. Pressure vessels
  - D. Stability, vibrations
  - E. Limitations on lamination theory

## TWO DIMENSIONAL PROBLEMS IN THE THEORY OF ANISOTROPIC ELASTICITY

### Recommended Textbooks:

N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff (1963)

S. G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Elastic Body*, Holden-Day (1963)

### Course Abstract:

The first half of the course is devoted to the formulation and solution of the two dimensional, isotropic elastic problem using complex variable methods. Solutions are obtained using the Laurent series expansion for multiply-connected bodies. The second half of the course is devoted to the analysis of anisotropic, two dimensional problems, again using the complex variable method. Example problems and projects are chosen for their relevancy to current engineering problems in anisotropic media, such as advanced fiber composites. Existing numerical solution methods such as finite elements and integral equations are used and compared to the analytic results when possible. The course assumes a knowledge of the basic theorems of analysis of functions of a complex variable as well as the basic theory of elasticity.



## TWO DIMENSIONAL PROBLEMS IN THE THEORY OF ANISOTROPIC ELASTICITY

### COURSE OUTLINE:

#### I. Review of complex variable theory (6 hours)

- A. Analytic functions
- B. Green's theorem
- C. Cauchy integral theorems
- D. Series

#### II. Plane theory of isotropic elasticity (18 hours)

- A. Equilibrium; stress function
- B. Strains; Hooke's law
- C. Goursat formula
- D. Displacements
- E. Traction
- F. Kolosov formula
- G. Forces on a contour
- H. Single-valued displacements, stresses
- I. Laurent series for the stress functions
- J. Infinite region with a hole
- K. Polar coordinate form of the equations
- L. Mapping functions; curvilinear coordinates
- M. Transformed field equations
- N. Example solutions

#### III. Plane theory of anisotropic elasticity (15 hours)

- A. Hooke's law for various types of anisotropy
- B. Stress function
- C. Characteristic surfaces for the stress function
- D. Roots of the characteristic equation  $\ell_4(u) = 0$
- E. Stresses and displacements
- F. Forces on a contour
- G. Infinite region with a hole
- H. Single-valued stresses and displacements
- I. Mapping functions
- J. Fourier analysis of the boundary conditions
- K. General expansion form of the solution
- L. Example solutions



## 1.7 APPENDIX II: TRIP REPORTS

TRIP REPORT NO.	TITLE	DATE
TR-69-02	Exploration of Possible University-Industry Cooperation in the Area of Advanced Composite Technology (T. A. Cruse)	7/14/69
TR-69-04	Detailed Discussion of Proposed University-Industry Joint Program in Advanced Composite Technology (T. A. Cruse)	8/11-12/69
TR-69-09	Advanced Composites Status Review (T. A. Cruse)	9/30-69 10/2-
TR-69-10	University Team Visit to Air Force Materials Laboratory	11/24/69
TR-70-01	Fuselage Program Review (General Dynamics) and Discussion of Project Problems (T. A. Cruse)	1/7-9/70
TR-70-02	Review Meeting, First Edition of Structural Design Guide for Advanced Composite Applications, and Test Methods (R. D. Blevins)	2/11/70
TR-70-03	Discussion of Bolt Bearing Testing Procedures with North American Rockwell/Columbus (J. P. Waszczak)	3/12/70
TR-70-04	Discussion of Test Data, Methods with North American Rockwell/Los Angeles (R. D. Blevins)	3/18/70
TR-70-05	Team Visit to Southwest Research Institute	4/2/70
TR-70-09	Team Visit to General Dynamics/Fort Worth	4/3/70
TR-70-10	Discussion of Consulting Program with Dr. Frank J. Rizzo (T. A. Cruse)	8/19-21/70
TR-70-12a	Project Review Meetings at General Dynamics/Fort Worth and Air Force Materials Laboratory	10/4-6/70

<i>TRIP REPORT NO.</i>	<i>TITLE</i>	<i>DATE</i>
TR-70-13	Boeing/Vertol: Review of Boron Blade Program (S. J. Marulis; T. A. Cruse)	10/28/70
TR-70-14	NASA/Langley Field; Interactive Program in Composites at CMU (T. A. Cruse, J. L. Swedlow)	12/15/70
TR-71-01	AFML; GD/Ft. Worth: Program Review and New Project Proposals (T. A. Cruse)	1/5-6/71
TR-71-02	GD/Ft. Worth: Program Review Meeting (T. A. Cruse)	4/14/71
TR-71-03	Fifth St. Louis Symposium on Composite Materials	4/6-7/71
	12th AIAA/ASME Structures, Structural Dynamics, and Materials Conference (T. A. Cruse, J. P. Waszczak)	4/19-21/71
TR-71-06	Design Guide Review Meeting; NAR, Los Angeles (T. A. Cruse)	5/24-26/71
TR-71-07	GD/Ft. Worth; Program Review Meeting (T. A. Cruse, J. P. Waszczak, H. J. Konish, Jr.)	6/9-10/71
TR-71-08	Boeing/Vertol: Review of CMU Fracture program (H. J. Konish, Jr.)	6/18/71
TR-71-09	31st National Applied Mechanics Conference (T. A. Cruse)	6/23-25/71
TR-71-10	GD/Ft. Worth: Review of Summer Project (J. P. Waszczak)	7/7-9/71
TR-71-11	GD/Ft. Worth: Review of Summer Project, Boundary-Integral Project (J. P. Waszczak, T. A. Cruse)	8/5/71
TR-71-12	GD/Ft. Worth: Review of Summer Project, (J. P. Waszczak)	8/5-6/71
TR-71 13	5th National Fracture Mechanics Symposium (H. J. Konish, Jr., T. A. Cruse, J. R. Gsias)	8/31-9/2/71

### 1.8 APPENDIX III: RESEARCH DOCUMENTS

REPORT NUMBER	TITLE	DATE
SM-22	Anisotropic Stress Strain Program Layer Usage Guide (H. J. Konish, Jr.)	January 1970
SM-23	Project Problems for Air Force Con- tract F33615-70-C-1146 (T. A. Cruse)	January 1970
SM-24	Summary of the Direct Potential Method (T. A. Cruse;	January 1970
SM-25	Interactive Program in Advanced Composites Technology (T. A. Cruse)	February 1970
SM-27	Symmetric Laminate Constitutive Equation Program-EMAT Usage Guide (J. P. Waszczak)	February 1970
SM-28	Bolt Bearing Specimen Co-ordinate Transformation Program - Usage Guide TRANS (J. P. Waszczak)	April 1970
SM-29	Certain Aspects of Design with Ad- vanced Fibrous Composites (R. D. Blevins)	April 1970
SM-31	Stress Analysis of a Cracked Ad- vanced Composite Beam (H. J. Konish, Jr.)	April 1970
SM-32	An Investigation of Fracture in Advanced Composites (W. H. Bamford)	April 1970
SM-34	An Investigation of Stress Concentra- tions Induced in Anisotropic Plates Loaded by Means of a Single Fastener Hole (J. P. Waszczak)	May 1970
SM-38	Integral Equation Methods in Potential Theory (T. A. Cruse)	August 1970
SM-41	Stress Analysis of a Cracked Aniso- tropic Beam (H. J. Konish, J. L. Swedlow)	September 1970
SM-42	An Investigation of Stress Concentra- tions Induced in Anisotropic Plates Loaded by Means of a Single Fastener Hole (J. P. Waszczak, T. A. Cruse)	September 1970
SM-45	The Use of Singular Integral Equations with Application to Problems of Composite Materials (F. J. Rizzo)	October 1970

REPORT NUMBER	TITLE	DATE
SM-49	Report on the Relation Between the Stiffness Matrix and the Angle of Rotation of a Lamina (J. Kolter)	November 1970
SM-50	Numerical Solution Accuracy for the Infinite Plate with a Cutout - Progress Report (W. Bamford)	December 1970
SM-52	Failure Mode and Strength Predictions of Anisotropic Bolt Bearing Specimens (J. P. Waszczak; T. A. Cruse)	September 1970
SM-53	A Proposed Experimental Investigation of Fracture Phenomena in Advanced Fiber Composite Materials (H. J. Konish, Jr.)	February 1971
SM-63	Loaded Circular Hole in an Anisotropic Plate (J. P. Waszczak)	May 1971
SM-64	Stress Analysis of the Crack-Tip Region in a Cracked Anisotropic Plate (H. J. Konish, Jr.)	May 1971
SM-65	Numerical Calculation of the Characteristic Directions for a Generally Anisotropic Plate - MULTMU Usage Guide (H. J. Konish, Jr.)	June 1971
SM-68	Solution to Kelvin's Problem for Planar Anisotropy (W. Bamford)	June 1971
SM-70	USER'S DOCUMENT: Two Dimensional Boundary-Integral Equation Program (T. A. Cruse)	June 1971
SM-71	Optimization of Advanced Composite Plates (S. Marulis)	June 1971
SM-72	Two Dimensional Anisotropic Boundary-Integral Equation Method (W. H. Bamford, T. A. Cruse)	August 1971

REPORT NUMBER	TITLE	DATE
SM-74	Experimental Investigation of Fracture in an Advanced Fiber Composite (H. J. Konish, J. L. Swedlow, T. A. Cruse)	September 1971
SM-76	Toward a Design Procedure for Mechanically Fastened Joints Made of Composite Materials (J. P. Waszczak)	September 1971
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## CHAPTER II

### FRACTURE OF ADVANCED COMPOSITES

#### 2.1 *STRESS ANALYSIS OF A CRACKED ANISOTROPIC BEAM*

##### 2.1.1 *Introduction*

The high specific strength and specific stiffness of advanced fiber composite materials have made them very attractive to the aerospace industry. The fact that they are both anisotropic and inhomogeneous, however, has somewhat retarded their use, as the design and analysis procedures developed for metals are not strictly applicable; thus, it is necessary to adapt old procedures, or develop new ones, which can deal with the more complex composite materials.

The project discussed in this chapter deals with one such effort. The specific problem under consideration is the effect of a crack in a unidirectional advanced fiber composite material. Although this problem is one of great significance in aerospace structures, it has not yet been extensively treated. An analytic solution has been derived for the elastic stresses and strains induced by a crack in a loaded anisotropic plate [1]. The solution does assume material homogeneity, but this is a good approximation for advanced fiber composite materials on a macroscopic scale. However, relatively little has been done to follow up the analytic work.

##### 2.1.2 *Review of Previous Work*

The most extensive investigation of fracture of composites in the literature is that done by Professor E. M. Wu of Washington University, St. Louis. He considers the problem of a central crack, aligned with the fibers of a unidirectional composite material, which are, in turn, aligned with the edges of a plate subjected to general edge loadings.

Wu demonstrates that linear-elastic fracture mechanics are applicable to this problem [2]. His analysis yields results of the form

$$\sigma = K_I F / \sqrt{2\pi r} \quad (1)$$

$$K_I = \sqrt{a} G \quad (2)$$

where  $F$  is a function of the external loading and  $G$  is a function of specimen geometry, material constants, and external loading. These results are similar in form to the results obtained from the analysis of an isotropic problem.

Wu verified his analysis experimentally [2,3]. His experimental work, (done with fiberglass plates), does demonstrate the applicability of a linear elastic fracture mechanics analysis to his particular problem. It further shows that the critical stress intensity factors  $K_{Ic}$  (corresponding to symmetric loading on the plate) and  $K_{IIc}$  (corresponding to skew-symmetric loading on the plate) are material constants. Under combined external loading, the following empirical relationship is observed to be valid at incipient unstable crack propagation:

$$K_I/K_{Ic} + (K_{II}/K_{IIc})^2 = 1 \quad (3)$$

This result is not, however, particularly surprising in view of [1], where it is analytically shown that any arbitrary two-dimensional fracture problem in an anisotropic material may be decomposed into two independent problems, one symmetric and one skew-symmetric. Thus, only the form of (3) may be considered as original; its existence is predicted by analysis.

Wu has also investigated the problem of an external loading of combined compression and shear [4]. This loading will lead to crack propagation by the second, or "sliding" mode. Three possible subcases

are considered analytically: Relative displacement of the crack surfaces, over a portion of the crack surfaces, and over none of the crack surfaces.

This analysis was verified by an experimental program carried out on fiberglass. The results show that, for ratios of compressive load to shear load greater than approximately 0.4, failure does not occur by unstable crack propagation; the crack velocity remains quasi-stable until the specimen fails from propagation of the crack completely through it. If the ratio of compressive load to shear load is increased, internal buckling of the fibers and separation of the fibers from the matrix is observed; at most, the crack will propagate some small distance at an angle of  $45^\circ$  from its initial direction, then diffuse and die out. The specimen buckles thereafter with no additional crack propagation. Wu thus concludes that fracture mechanics is only applicable to this problem when the ratio of compressive stress to shear stress is less than 0.4. The second subcase of the analysis gives the best agreement between analysis and experiment when fracture mechanics are applicable. The quasi-stable crack propagation found to occur experimentally when the ratio of compressive load to shear load is approximately 0.4 seems to be well-described by the first subcase of the analysis. The third subcase of the analysis is believed to be applicable when the compressive load is sufficiently large to prevent crack extension; however, buckling, rather than crack propagation, becomes the dominant mode of failure before this load is reached, so the presence of the crack not significant in the failure of the specimen.

Wu notes that stable crack propagation occurs in an intermittent manner in fiberglass [2]; he postulates that this is caused by the crack



crossing the reinforcing fibers. This hypothesis is investigated both analytically and experimentally [5].

The analysis is based on the assumption that crack growth is primarily caused by the component of tensile stress perpendicular to the direction of crack growth, as the intermittent stable crack propagation is most frequently observed under skew-symmetric loading. It indicates that the crack does not necessarily propagate in a direction collinear with itself, but rather at an angle where the combination of the size of a sub-critical flaw and the maximum tensile stress reaches some critical value, causing the flaw to grow. Under skew-symmetric loading, the maximum tensile stress is not perpendicular to the crack direction, and, assuming that flaws of any given size are uniformly distributed in the material, the crack will propagate at some angle to its initial direction. Since the initial direction of the crack is collinear with the fibers, the propagating crack must cross fibers. The direction of crack growth is thus a function of the direction of the shear loading.

It is noteworthy that Wu finds the Griffith energy criterion to be applicable to composite materials only when the crack propagates across fibers. Although Wu offers no explanation for this anomaly, it may be due to the fact that, for this particular geometry, the crack propagates only through resin unless it crosses fibers. Thus, the crack would "sense" a brittle, high-strength material, for which the Griffith criterion is applicable, only when it crosses fibers.

Wu's specimen is also analyzed for symmetric loading by Bowie and Freese [6]. They use a modified mapping-boundary collocation technique to derive the stress intensity factor numerically. Of particular interest

is the result of Bowie and Freese that, when the strength of the material in a direction transverse to the crack is much larger than the strength of the material collinear with the crack, the stress intensity factor is not longer the same for both the isotropic and anisotropic cases, as predicted by Sih, Paris, and Irwin [1]. However, Bowie and Freese do note that, when the strength of the material in the direction collinear with the crack is greater than or equal to the strength of the material in a direction transverse to the crack, the two stress-intensity factors agree to within five per cent.

### *2.1.3 Analytical Study*

The efforts described above comprise the significant work now available in open literature on macroscopic analysis of fracture in anisotropic materials. Both of them consider only cracks which are aligned with the fibers of the composite material, and must therefore be considered incomplete, as no provision has been made for cracks with arbitrary orientation to the material axes. The purpose of the project described in this section is to investigate the behavior of a crack in an anisotropic material where the crack is not, in general, collinear with one of the material axes (though these cases are considered). Information is also sought on the behavior of the stress-intensity factor as the orientation of the crack with respect to the material axes and the specimen geometry are varied. Finally, it is desired to obtain verification of either Bowie and Freese [6], or Sih, Paris, and Irwin [1] concerning the differences, if any, between the isotropic and anisotropic stress intensity factors.

In pursuit of these objectives, a series of anisotropic three-point bend specimens with edge cracks of different lengths (Fig. 1) has been studied analytically to determine the stress and deformation response in the vicinity of the crack-tip. Material properties were chosen such that the specimen represents uni-directional boron/epoxy. The orientation of the material axes relative to the crack-axis is completely arbitrary.

The analysis was performed using a linear elastic, plane stress, finite element technique. Two element grids were used, one representing the entire beam and the other representing a small region of the beam surrounding the crack-tip. The latter grid is used to provide more detailed information in the region of the crack-tip than can be obtained from the relatively coarse grid of the entire beam and still remain in the core of the computer. Details of the numerical studies are contained in [7].

Load is applied to the beam by specifying the transverse displacement of the point on the upper edge of the beam in line with the crack-axis. Appropriate nodal displacements from the grid of the full beam are then applied to the grid of the crack-tip region as boundary conditions. From the analysis of the grid of the crack-tip region, stresses and displacements are determined as functions of position.

The stresses and deformations are represented in the form given by Sih, Paris, and Irwin [1]:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[ \frac{\mu_1 \mu_2}{\mu_1 - \mu_2} \left( \frac{\mu_2}{\sqrt{\cos\theta + \mu_2 \sin\theta}} - \frac{\mu_1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} \right) \right] \quad (4)$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[ \frac{1}{\mu_1 - \mu_2} \left( \frac{\mu_1}{\sqrt{\cos\theta + \mu_2 \sin\theta}} - \frac{\mu_2}{\sqrt{\cos\theta + \mu_1 \sin\theta}} \right) \right] \quad (5)$$

$$v = K_I \frac{\sqrt{2r}}{\pi} \operatorname{Re} \left[ \frac{1}{\mu_1 - \mu_2} (\mu_1 q_2 \sqrt{\cos\theta + \mu_2 \sin\theta} - \mu_2 q_1 \sqrt{\cos\theta + \mu_1 \sin\theta}) \right] \quad (6)$$

where  $K_I$  is the stress intensity factor for an isotropic specimen of the same geometry as that being analyzed;  $r$  and  $\theta$  are the coordinates shown in Figure 1. The  $\mu_j$  are the roots of the characteristic equation

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{16} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0 \quad (7)$$

where  $a_{ij}$  are the material compliances as given by

$$\epsilon_i = a_{ij}\sigma_j \quad (8)$$

The  $q_j$  are defined as

$$q_j = a_{12}\mu_j + a_{22}/\mu_j - a_{26} \quad (9)$$

Using the equations (4-9), the stress intensity factor  $K_I$  can be obtained in various ways from both the stresses and the displacements found in the analysis of the crack-tip region. It is hypothesized that the stress intensity factor is a separable function of the load on the beam and the specimen geometry, i.e.,

$$K_I = f(\text{load}) g(\text{geometry}) \quad (10)$$

Since the analysis is linear elastic,

$$f(\text{load}) = P/B \quad (11)$$

The effect of the specimen geometry is a function of the crack-length, the effects of finite specimen boundaries, and possibly the material anisotropy. It is further hypothesized that

$$g(\text{geometry}) = \sqrt{a} G(a/W, \alpha) \quad (12)$$

where the function  $G$  contains the effects of the finite boundaries of the specimen and any effect of the material anisotropy. Thus, combining equations (10-12)

$$K_I = (P\sqrt{a} / B) G(a/W, \alpha)$$

or

(13)

$$G(a/W, \alpha) = K_I B / P\sqrt{a}$$

The function  $G(a/W, \alpha)$  has been obtained analytically for three values of  $\alpha$  and five values of  $a/W$ , using values of  $K_I$  obtained from both stress and displacement data. Each  $G(a/W, \alpha)$  was then normalized on the value  $G(0.2, \alpha)$  for corresponding methods of determining  $K_I$ . The resulting value, denoted as  $\bar{G}(a/W, \alpha)$  is shown plotted as function of  $a/W$  in Figure 2. On the same graph is shown a curve representing  $\bar{G}(a/W)$  for an isotropic specimen, as obtained from [8]. The data points show satisfactory agreement with the curve, in view of the numerical noise introduced by two finite element grids which are not entirely compatible. Thus,  $\bar{G}(a/W, \alpha)$  is identical with  $\bar{G}(a/W)$ . This, in turn, implies that the anisotropic stress intensity factor is the isotropic stress intensity factor.

Although the stress intensity factor in equations (4-6) is the isotropic stress intensity factor, stress and deformation are functions of material constants. Thus, fracture in advanced fiber composite materials cannot be ascribed solely to any combination of the stress intensity factors. To some extent, therefore, the applicability of fracture mechanics to composite materials is questionable. Exactly what importance a crack has in composite materials, and what role the material properties play in describing it, are questions which were investigated experimentally and are reported in Section 2.2.

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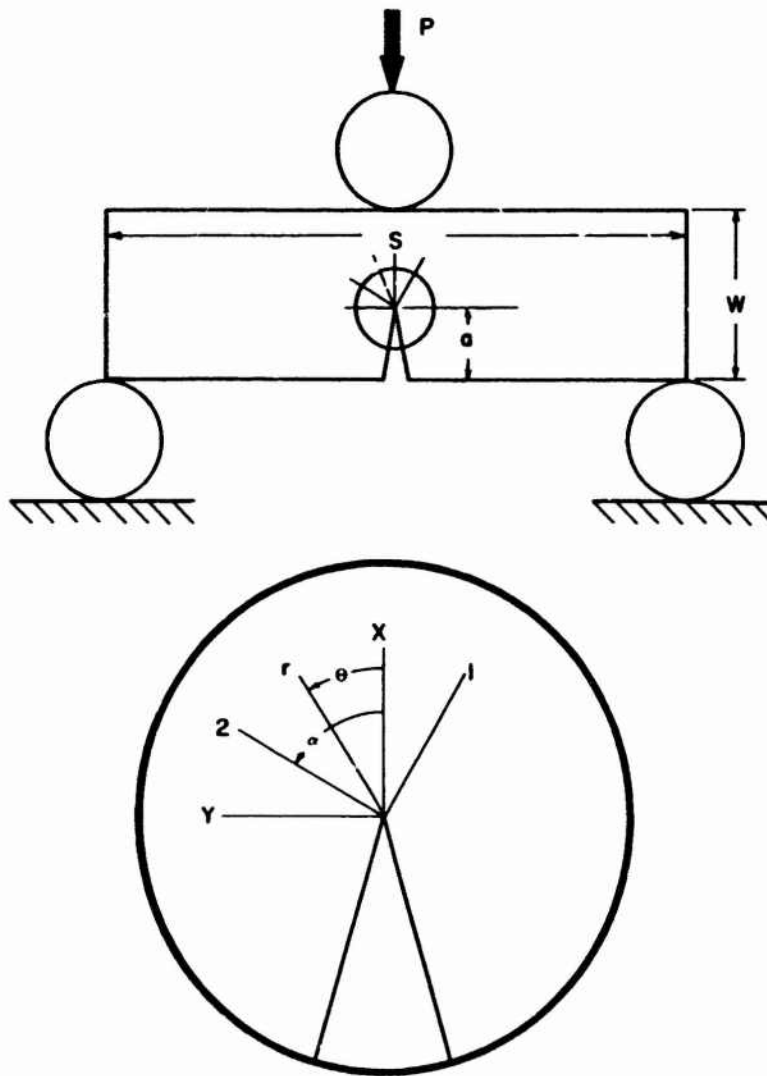


Figure 1: Three-point bend fracture specimen, with global ( $x, y$  and  $r, \theta$ ) and material ( $1, 2$ ) coordinate systems shown (insert). The applied load  $P$  is modelled as point load. The specimen thickness is denoted by  $B$ .

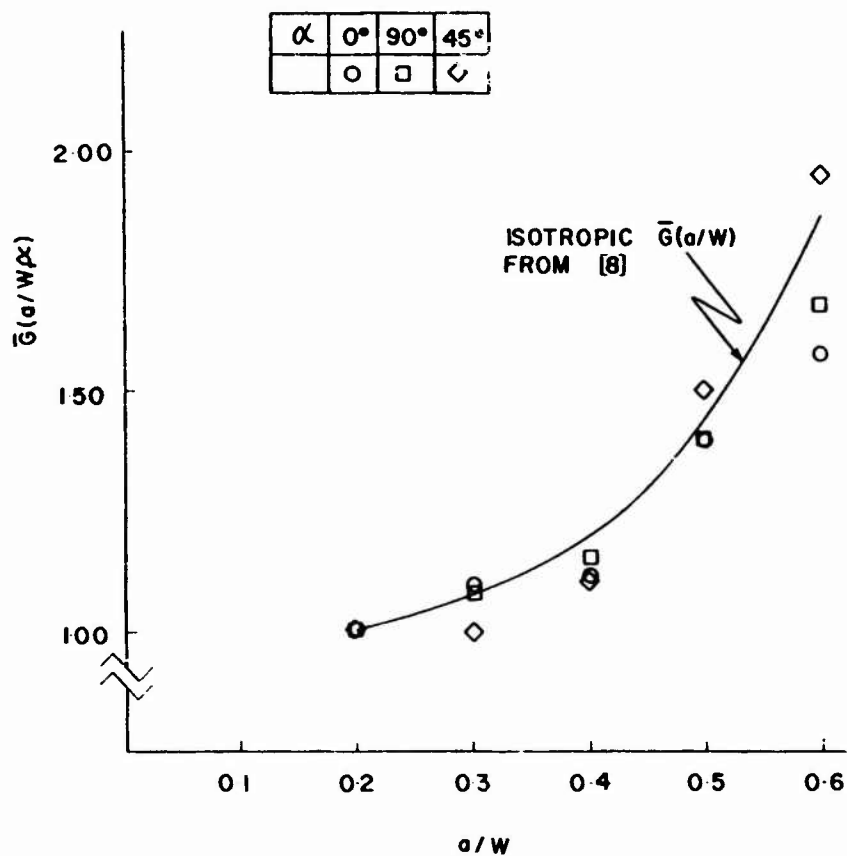


Figure 2: A plot of  $\bar{G}(a/W, \alpha)$  vs.  $a/W$ . The degree of correspondence between the discrete points (obtained numerically) and the continuous curve (obtained from [8]) is a measure of the applicability of an anisotropic continuum analysis [4] to advanced fiber composite materials.



## 2.2 *EXPERIMENTAL INVESTIGATION OF FRACTURE IN AN ADVANCED FIBER COMPOSITE*

### 2.2.1 *Introduction*

Linear elastic fracture mechanics (LEFM) is now accepted as the rationale for characterizing crack toughness of materials that are ostensibly homogeneous and isotropic, the outstanding examples being a wide range of metallic alloys. The basic experience that supports this approach is that presence of a macro crack dominates the response of a structure to remote loading. With the advent of advanced fiber composites, however, there arises the question of the degree of homogeneity of the structure surrounding the crack that is necessary for LEFM to be applicable. In particular, there is concern over whether heterogeneity and anisotropy will preclude practical use of LEFM in composites.

Vigorous discussion of this issue is important and widespread, but the interchanges so far have tended to be theoretical and even speculative. In an effort to supply some physically based information, a pilot series of experiments has been performed, to answer two specific questions:

1. If a cracked, composite specimen is loaded to failure, is the path of crack prolongation determined by the geometry of the initial crack and the loading, or by material orientation?
2. Can LEFM, suitably modified to account for material anisotropy, be usefully applied to composites?

The data now in hand, although limited, indicates that a crack in a composite is at least influential in determining failure patterns and, in many cases, the crack is dominant; furthermore that LEFM provides useful

procedures for evaluating crack toughness of composites.

This section gives a brief review of the test procedures, methods of data reduction, and experimental results. Observations made during the course of the tests are reported, and failure surfaces are shown. Analytical work stimulated by these results is underway and will be reported subsequently.

#### *2.2.2 Test Procedures; Program*

It was obvious from the objective of the test program that the test procedures should follow those developed within the framework of conventional fracture mechanics. There is, in fact, a wealth of literature on this subject including an ASTM Tentative Method [1] and extensive interpretation of it (see, e.g., [2,3]). Departures from the specifications in [1] were minimal and were dictated either by the special nature of the material under test or by simple practicality.

The three-point bend specimen prescribed in [1] was chosen largely to bypass problems associated with gripping the test piece. (See Figure 1.) In the extensive data base that now exists for metals testing, results for this configuration compare well to those for other geometries so that, among other matters, there was no reason to expect that the bearing load opposite the crack front should influence unduly the processes of crack prolongation. In fact, the data reduction scheme in [1] accounts for such details of specimen geometry and load arrangement.

The specimen proportions shown in Figure 1 follow the recommendations in [1] except that the crack front was not sharpened under fatigue loading. Instead, the notch was produced by a sawcut followed by a final lengthening and sharpening using an ultrasonic cutter.

As shown in Figure 2, each specimen was centered on two parallel rollers (1 in dia) whose centerlines were 4 in apart. A third parallel roller was then located directly above the crack, and the specimen was loaded vertically downward. Testing was performed in an Instron machine of 10,000 lb capacity, and cross-head motion was set at  $10^{-2}$  in/min to minimize dynamic effects. Load and cross-head motion were monitored during each test and then cross-plotted to give the basic data for later reduction. While the requirement of [1] is to record crack-mouth opening by means of a special clip gauge, both the basic linearity of material response and the rigidity of the test machine, relative to the specimen, seemed to make this degree of fidelity to [1] unnecessary for the pilot test series.

The program involved twenty-three specimens, thus allowing for two reproducibility tests, and for the testing of both uni- and multi-directional laminates having a range of starter crack lengths. The material used was a NARMCO graphite-epoxy with Morganite II fibers in 5206 resin.

Reproducibility was evaluated by testing two sets of five specimens, each set of the same lay-up and geometry. The first set was a uni-directional laminate ( $\alpha = 0^\circ$ ) and had an initial crack length of 0.4 in. The second set was multi-directional ( $\alpha = (0^\circ/\pm 45^\circ/90^\circ)_S$ ) and had the same starter crack length. Single tests were run for  $\alpha = 0^\circ, 45^\circ, 90^\circ; (\pm 45^\circ)_S$ ; and  $(0^\circ/\pm 45^\circ/90^\circ)_S$ . Starter crack lengths were 0.2, 0.4, and 0.6 in, the shortest of which was less than the requirements in [1]. Such specimens were included to permit evidence of material dominance to develop.

### 2.2.3 Data Reduction; Results

A typical load-cross-head displacement trace is reproduced in Figure 3. There is an initial region of increasing slope during which slack in the load train is taken up, and bearing surfaces under the loading rollers develop. This is followed by a linear region in which the specimen deforms elastically. A third region of decreasing slope then begins as a result both of nonlinear load-displacement behavior and of damage initiation. Finally the load peaks and falls off as the test piece breaks in two.

In order to differentiate the nonlinear effects from those ascribable to damage, the Tentative Method prescribes the following data reduction scheme.<sup>1</sup> The slope  $M_0$  of the linear portion of the curve is identified, and a line of slope 5% less than  $M_0$  is drawn as shown in Figure 3. This line intersects the curve at a load termed  $P_S$ . If  $P_S$  is the greatest load withstood by the specimen to that point in the test,  $P_S$  is set equal to  $P_Q$ . If any load maximum precedes  $P_S$ , then  $P_Q$  is equated to that maximum value. In either case, the experience in metals testing has shown  $P_Q$  to correspond reasonably well to the point of failure initiation. In the absence of a suitable data base for composites, this procedure was used to find  $P_Q$ ; the data obtained is thus surely consistent and probably conservative. Together with specimen geometry,  $P_Q$  is then used to compute  $K_Q$ , the critical stress intensity or candidate fracture

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<sup>1</sup>It should be borne in mind that the present discussion is but an abstract of a most explicitly defined procedure; the interested reader is urged to consult [1] for complete details.

toughness.<sup>2</sup> See [2,3].

For each laminate, the  $K_Q$  values were averaged to give  $\bar{K}_Q$  which, in turn, was used to find a critical strain energy release rate  $G_Q$  — see [3,4]. The results are shown in Table I. Also of interest are the failure surfaces, depicted in Figures 4-8; a specimen that did not part fully is shown in Figure 9.

#### 2.2.4 Discussion

At the outset, two questions were posed regarding the utility of LEFM in characterizing fracture of composites. The first concerns paths of crack prolongation; the answer may be inferred from inspection of the failure surface. The second involves use of LEFM as a data reduction scheme; the answer to this question comes from physical measurements.

The appearance of the failure surfaces suggests that, in the main, the crack and loading dominate fracture. In Figure 4 (specimens for which  $\alpha = 0^\circ$ ), the path of crack growth is observed to be roughly coplanar with the starter crack. Note that in the case of the longest crack ( $a = 0.6$  in), where a longitudinal secondary crack formed, the path is generally forward. Indeed, the crack seems to have made a series of sharp turns to regain its coplanar path.

It is not surprising, on the other hand, to see that, in the  $\alpha = 45^\circ$  specimens, the crack grew along a plane containing no fibers. This is clear in Figure 5 and, although fracture occurred as the result

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<sup>2</sup>In metals testing, certain additional steps are taken to establish the validity of an individual test result. Since these steps necessitate use of the yield stress, they cannot be followed in this work. Thus only *candidate* values of fracture toughness, or  $K_Q$ , are reported. The data cannot be presumed to give  $K_{Ic}$  for these materials because compliance with the strict requirements of [1] are definitionally impossible.

of crack propagation (in the matrix), the mode is a mixture of opening and sliding [3]. More sophisticated instrumentation would have permitted articulation of the relative presence of each mode, but such instrumentation was not used in this program.

Forward crack growth is evident for the  $\alpha = 90^\circ$  specimens as depicted in Figure 6. Growth again was along a plane containing no fibers which, in this case, is coplanar with the starter crack.

During testing the uni-directional specimens described above emitted popping noises prior to failure. Because the fracture process also involved matrix breaking of one sort or another, the two phenomena are believed to be related. Even in the  $\alpha = 0^\circ$  specimens, the crack appears at the outset to have operated on virtually independent fiber bundles as they pulled out from the matrix. The resulting failure surfaces are very rough for the early stages of growth but then become more nearly uniform. The noise levels for the remaining specimens were much lower, and their failure surfaces are less suggestive of matrix cracking.

Figure 7 is instructive in that it shows for the  $\alpha = (\pm 45^\circ)_S$  test pieces an increasing crack dominances as the starter crack is made longer. For  $a = 0.2$  in, the crack path almost immediately turns  $45^\circ$  from its initial orientation, there being but a slight indication of forward growth. A greater tendency toward coplanar growth is apparent when  $a = 0.4$  in, and crack dominance is manifest when  $a = 0.6$  in. Crack growth is not possible on a plane containing no fibers — there being none by virtue of the lay-up — and some zig-zagging is apparent. This group of specimens thus shows a transition from some material dominance where the starter

crack is shorter than required by the Tentative Method to a fracture pattern fully dominated by the starter crack, as the length of the starter crack occurs.

Crack dominance is also clear in Figure 8, which shows failure surfaces for  $\alpha = (0^\circ/\pm 45^\circ/90^\circ)_S$ . In these specimens, the crack moved in its own plane but apparently grew further in the interior of the test piece than on its surface. An indication of this behavior, not uncommon in metals testing, is shown in Figure 9.

The use of  $K_Q$  to characterize behavior of these specimens appears, on the whole, to be warranted. The reproducibility tests on the  $\alpha = 0^\circ$  specimens<sup>3</sup> and the  $\alpha = (0^\circ/\pm 45^\circ/90^\circ)_S$  specimens were satisfactory. Load-displacement traces are shown in Figures 10 and 11, and the average  $K_Q$  values found are

$$\begin{aligned} \alpha = 0^\circ & : K_Q = 28.8 \times 10^3 \text{ lb/in}^2/\text{in} \quad \begin{matrix} +0.4\% \\ -5.5\% \end{matrix} \\ \alpha = (0^\circ/\pm 45^\circ/90^\circ)_S & : K_Q = 21.7 \times 10^3 \text{ lb/in}^2/\text{in} \quad \begin{matrix} +1.7\% \\ -3.2\% \end{matrix} \end{aligned}$$

The scatter is not unlike that found in metals testing. For three laminates, the data are fairly consistent with values obtained independently by Halpin [5] ( $25\text{--}28 \times 10^3 \text{ lb/in}^2/\text{in}$ ,  $\alpha = (0^\circ/\pm 45^\circ/90^\circ)_S$ ) and by Weiss [6] ( $31 \times 10^3 \text{ lb/in}^2/\text{in}$ ,  $\alpha = 0^\circ$ ;  $19 \times 10^3 \text{ lb/in}^2/\text{in}$ ,  $\alpha = (\pm 45^\circ)_S$ ) using other specimen geometries (shape and thickness) and load arrangements.

Inspection of Table I will show further that the  $K_Q$  values for various starter crack sizes are within a reasonable range of the average  $\bar{K}_Q$  for each laminate. It should also be noted that the majority of largest deviations occur for subsize starter cracks, and none of these is serious.

<sup>3</sup>One exception occurred for the  $\alpha = 0^\circ$  specimen set; because it was the first specimen of the entire series tested, it is presumably due to lack of experience with the test procedure, rather than material variation.

### 2.2.5 Conclusions

This pilot test series has been successful, for it has answered the questions posed at the outset. The failure mechanism of the specimen tested is crack dominated in most cases, and the procedures of LEFM can be applied even where the overt failure mechanism is not so obviously dominated by the starter crack.

There remains, however, a variety of questions about cracks in an advanced fiber composite material. Some concern the effects of specimen geometry and load arrangement, and can be answered only by further testing. Such work is needed, first, to define and delineate more fully the respective influence of cracks and material. Further, the entire matter of fracture in composites needs for its resolution an extensive data base similar to that which has evolved for metals. The building of this kind of experience is important not only to determine what constitutes meaningful laboratory work, but also to provide guidance in treating service situations. Experimentally determined  $K_Q$  values for given laminates might also be related to the properties of individual laminae within other laminates. Ultimately, the designer should be in a position to use fracture toughness as he would other material properties.

It would now appear that efforts to address these questions are warranted, for the present test series indicates that, when suitably modified to account for anisotropy, linear elastic fracture mechanics may usefully be applied to advanced fiber composite materials.



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TABLE I  
EXPERIMENTAL RESULTS

	$K_Q, \text{ lb/in}^2/\text{in} \times 10^3$			$\bar{K}_Q, \text{ lb/in}^2/\text{in} \times 10^3$	$G_Q, \text{ in lb/in}^2$
fiber orientation angle, $\alpha$	$a = 0.2 \text{ in}$	$a = 0.4 \text{ in}$	$a = 0.6 \text{ in}$		
0°	— 1	28.8	36.3	32.6 +11% -11%	117.
90°	1.66	1.46	— 2	1.56 +6.3% -6.3%	0.943
45°	0.690 3	2.22	2.39	2.30 +3.9% -3.8%	— 4
( $\pm 45^\circ$ ) <sub>s</sub>	18.5	18.5	16.3	17.7 +4.8% -9.4%	45.0
(0°/ $\pm 45^\circ$ /90°) <sub>s</sub>	23.5	21.7	20.5	21.9 +7.3% -8.6%	55.1

Notes:

1. Specimen was crushed before crack propagation occurred.
2. Instrumentation failure.
3. This value omitted when calculating  $\bar{K}_Q$ .
4. No  $G_Q$  available because the crack propagated in a mixed mode, which could not be directly uncoupled.

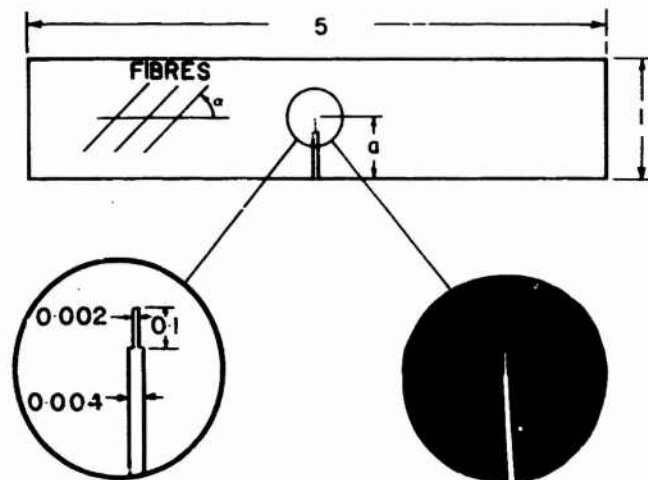


Figure 1: Three-point bend specimen geometry, with crack shape shown in inserts, both schematic (left) and actual (right). Fiber direction given by  $\alpha$ , crack length by  $a$ . Specimen thickness 0.5 in (nom); all dimensions given in inches.



Figure 2: Test jig at beginning of loading.

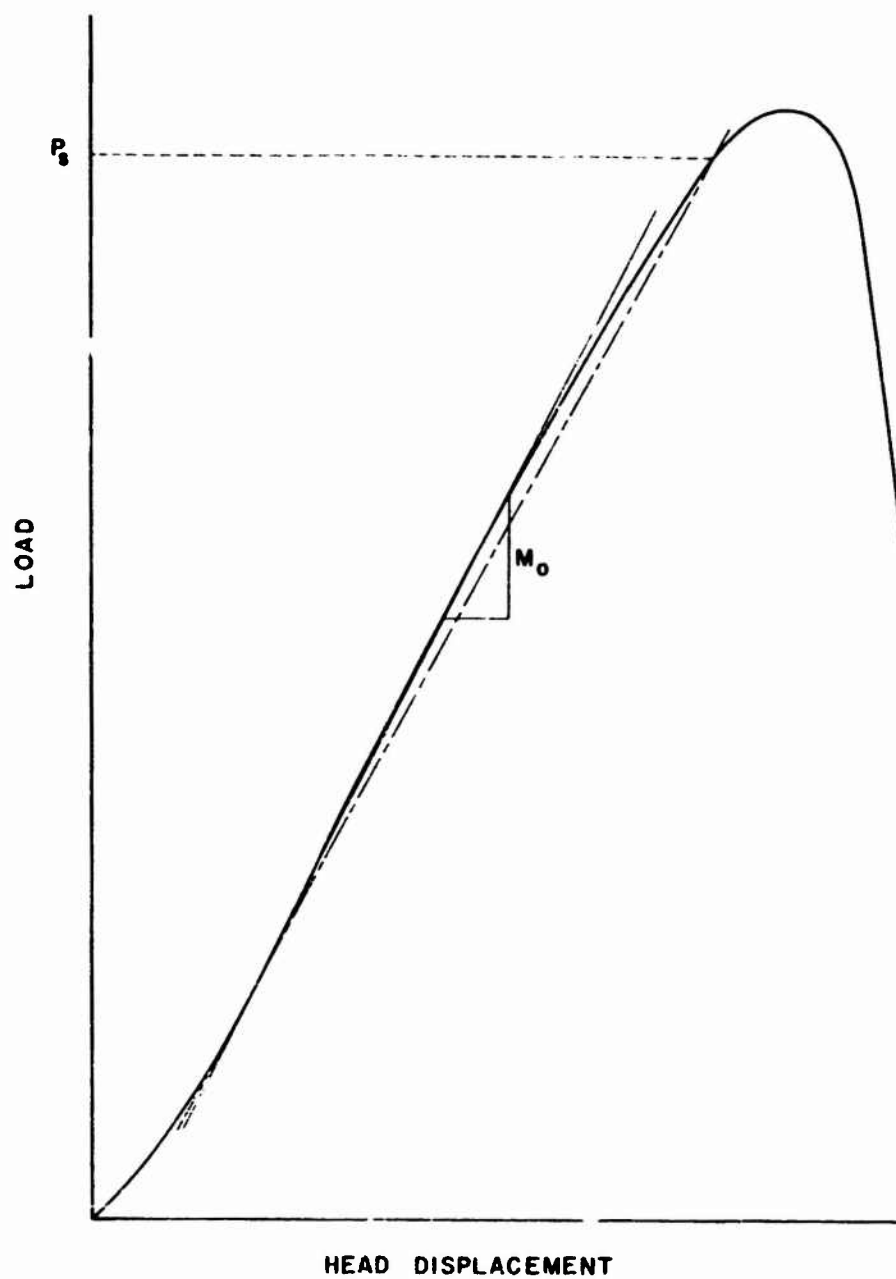


Figure 3: Typical trace of load applied to specimen vs. cross-head displacement, showing method used to determine  $P_s$ .



Figure 4: Failure surfaces for  $\alpha = 0^\circ$  specimens of three starter crack lengths ( $a = 0.6, 0.4, 0.2$  in).



Figure 5: Failure surfaces for  $\alpha = 45^\circ$  specimens of three starter crack lengths ( $a = 0.2, 0.4, 0.6$  in).

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best available copy.



Figure 6: Failure surfaces for  $\alpha = 90^\circ$  specimens of three starter crack lengths ( $a = 0.6, 0.4, 0.2$  in).





Reproduced from  
best available copy.

Figure 7: Failure surfaces for  $\alpha = (\pm 45^\circ)_s$  specimens of three starter crack lengths ( $a = 0.6, 0.4, 0.2$  in).



Figure 8: Failure surfaces for  $\alpha = (0^\circ/\pm 45^\circ/90^\circ)_S$  specimens of two starter crack lengths ( $a = 0.6, 0.4$  in).



Figure 9: Failed but unbroken specimen ( $\alpha = (0^\circ/\pm 45^\circ/90^\circ)_S$ ,  $a = 0.2$  in).

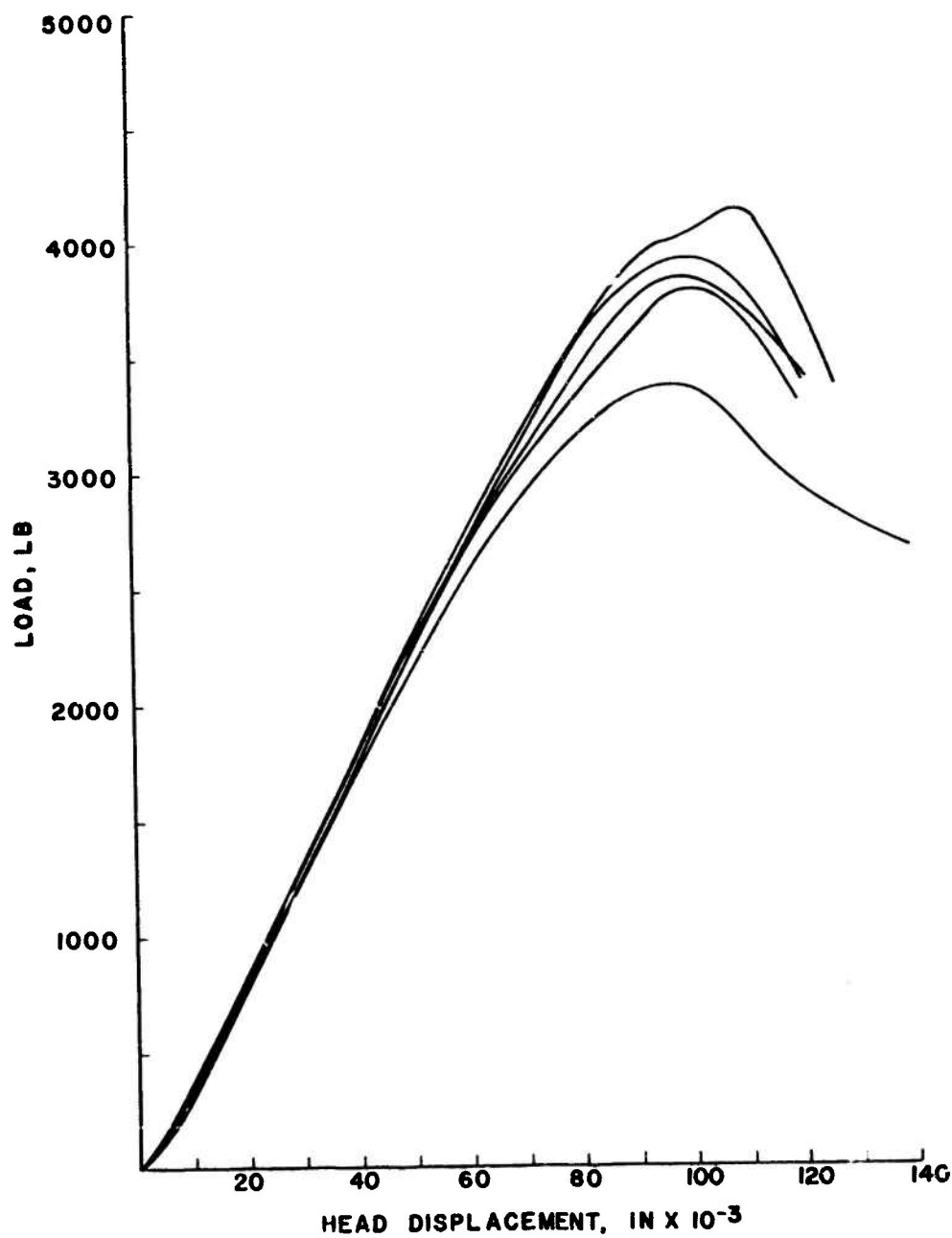


Figure 10: Traces of load vs. cross-head displacement for five specimens used in reproducibility tests for a uni-directional laminate ( $\alpha = 0^\circ$ ,  $a = 0.4$  in):

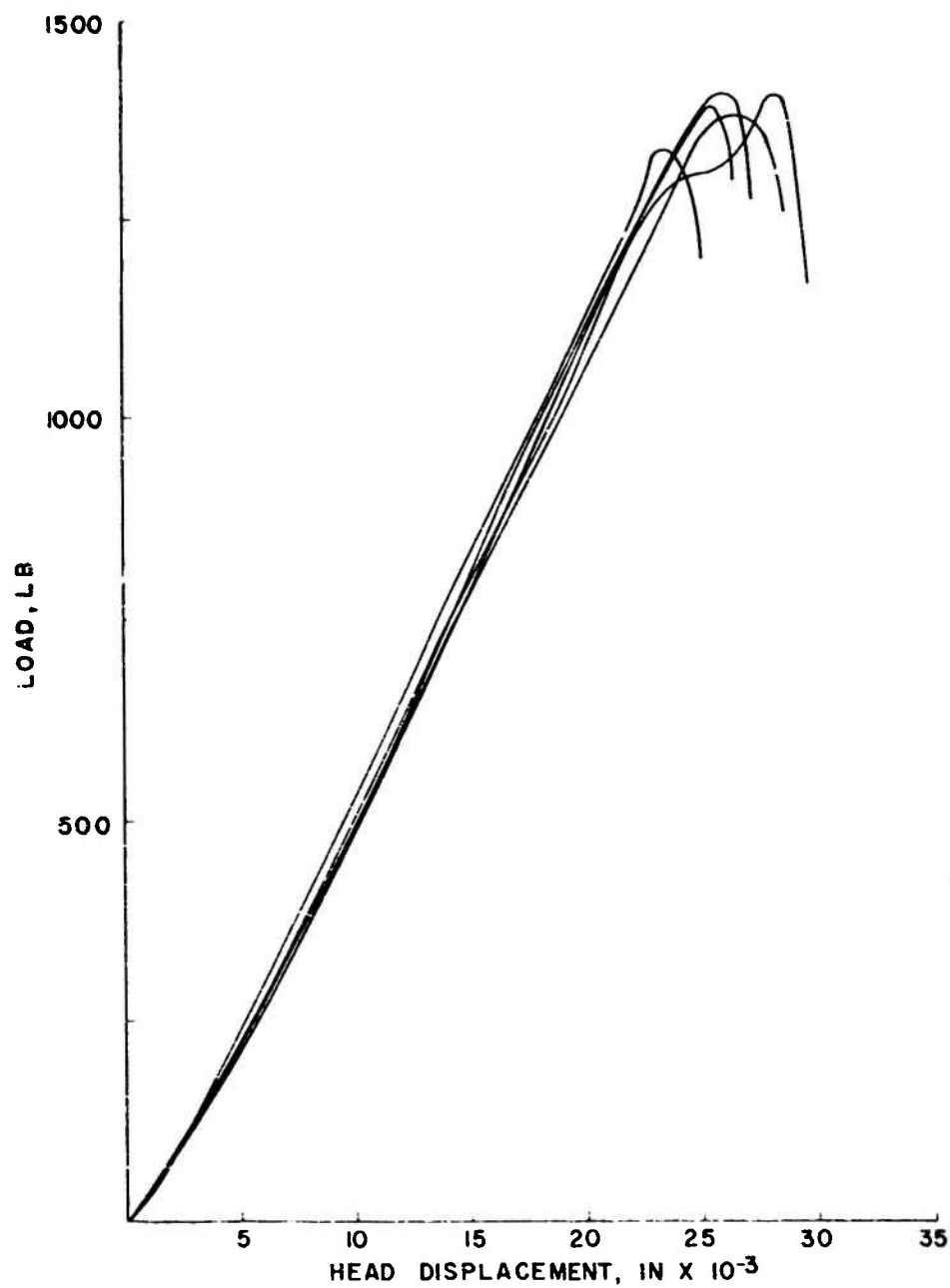


Figure 11: Traces of load vs. cross-head displacement for five specimens used in reproducibility tests for a multi-directional laminate ( $\alpha = (0^\circ/\pm 45^\circ/90^\circ)_s$ ,  $a = 0.4$  in).

## CHAPTER III

### STRENGTH OF MECHANICALLY FASTENED JOINTS

#### 3.1 *AN INVESTIGATION OF STRESS CONCENTRATIONS INDUCED IN COMPOSITE BOLT BEARING SPECIMENS*

##### 3.1.1 *Introduction*

This study is concerned with materials which consist of parallel, high strength fibers supported in a relatively ductile matrix material. The fibers act as load carriers while the matrix serves principally as a load transfer medium. In particular, it is concerned with advanced fibers, such as boron or graphite, in an epoxy matrix.

Because of their superior specific strength and specific stiffness, advanced fiber composite materials have a vast potential in the aerospace industry. Lamina, which are single layers of parallel fibers surrounded by the matrix material, are stacked at various orientations relative to one another to form a laminate. This procedure enables the designer to achieve desired strength and stiffness properties and to increase the structural efficiency of a given amount of material.

The strength and stiffness properties, however, are highly directional; panels fabricated out of layers of unidirectional composite tape are anisotropic. The designer therefore has the difficulty of including the effects of this anisotropy in his calculations.

One particular problem area in a structure made of composite materials is the bolted joint. The bolted joint in a composite material has a significantly lower efficiency than the same joint in metals. Furthermore, the composite joint may fail in unique modes not found in metal joints.

This study, therefore, investigates the stress concentrations induced in anisotropic plates loaded by means of a single fastener hole. The study is an attempt to further understand the failure characteristics of such bolted joints. The development of a prediction capability for both the failure mode and ultimate load is the major goal of the early part of this work. Such a capability would allow synthesis rather than analysis to be used in the future design of fastener joints. An implied goal in this study is an evaluation of the three proposed anisotropic failure criterion; maximum stress, maximum strain, and distortional energy.

### *3.1.2 Analysis Method*

A constant strain, finite element computer program using triangular elements was modified to handle anisotropic composite materials using lamination theory as presented in [1]. The experimental work done on bolt bearing specimens, from which this study draws heavily, only considered cross-plyed laminates which were mid-plane symmetric. As a result, this numerical study is also limited to this class of laminates. It is important to remember that the use of lamination theory ignores inter-laminar shear; consequently, it is expected that the degree of error in the results will vary with specimen anisotropy.

The design of a finite element grid representation to simulate the bolt bearing test specimen was subject to two major considerations. First, the grid had to be sufficiently detailed around the bolt hole to pick up the large stress gradients which are induced in this area. Secondly, the number of elements and nodes was restricted by the storage

capacity of the computer. Taking advantage of the two lines of specimen symmetry shown in Figure 1 only one-fourth of the specimen was included in the finite element simulation. Figure 2 shows a computer plot of the specimen section for  $e/d = 5.0$ ,  $s/d = 10.0$ , and  $\ell/d = 20.0$ . The grid representation used contains 480 elements and 279 nodes. The conditions of specimen symmetry are met by forcing the x-displacement of the vertical line of symmetry and the y-displacement of the horizontal line of symmetry to be zero in each computer run. A computer subroutine was also developed which transforms the co-ordinates of the grid shown in Figure 2 to any desired specimen geometry, i.e.,  $e/d$ ,  $s/d$ ,  $\ell/d$ .

To check whether or not the grid was sufficiently detailed around the hole an isotropic test case and several anisotropic test cases were run. A uniform tension stress was applied to the ends of each specimen. Comparison with the isotropic results presented in [2] (See Figures 3a and 3b) and the anisotropic results of [3] indicated that further refinement of the finite element mesh around the hole was not necessary. The observation that the computed finite element values of stress are higher than the exact values agree well with the results illustrated in [4].

A cosine distribution of normal stress acting over the upper half of the hole surface was used to simulate the resulting stress distribution caused by the bolt. The interaction was, therefore, assumed to be frictionless. Bickley [5] shows this to be an excellent approximation for isotropic bolt bearing specimens. A finite element analysis of the bolt-specimen interaction in certain composite laminates was performed at General Dynamics [6]. The cosine distribution of normal stress was again shown to be a realistic approximation of the interaction stresses.

Further confidence was gained in both the cosine distribution and the grid mesh by running an isotropic bolt bearing test problem and observing the qualitative agreement of the computed stress field around the hole surface (See Figure 4) with work by Coker and Filon [7]. The specimen used in their study had significantly larger values of  $e/d$  and  $s/d$  and thus a quantitative comparison was not possible.

Finally, two other normal distributions of stress, which were significantly different from the cosine distribution (See Figure 5), were used as the bolt-specimen interface stress boundary condition for one of the composite material specimen runs.

The net force in the load direction in each case was equivalent. It was observed that significant variance about the cosine distribution resulted in insignificant alterations of the calculated stress fields for the specimen considered.

### *3.1.3 Strength and Failure Mode Predictions*

The selection of specimen geometries for this investigation was made from data which has been published by General Dynamics [8,9] and Grumman Aerospace [10]. Included were two net-tension failure specimens, two shear-out failure specimens, one bearing failure specimen and one specimen which exhibited failures in a transition region between a net tension and combination failure mode. See Figure 6 for illustrations of these various failure modes.

Performing a strength analysis on a laminated composite material may be based on the strengths of its individual lamina. The strength of a single orthotropic lamina can, in theory, first be determined experimentally, producing an ultimate strength envelope for that material. This



three dimensional surface (in terms of principal lamina stresses) could then be used to analytically predict the ultimate strength of the total laminate. The state-of-the-art has yet to reach this level of sophistication. The present three dimensional ultimate strength envelope is constructed using only five points on the stress axes due to the, as yet, unsolved problems encountered in off-axis testing.

The Hill failure criterion is a widely accepted representation of this three dimensional envelope; it has been found in this study to be the only reliable means of predicting bolt bearing specimen failure modes. As shown in [11] lamina failure is predicted to occur when the following set of principal stress ratios (normalized on their respective ultimate stresses) add to a number, DIST, greater than or equal to one.

$$\text{DIST} = \left( \frac{\sigma_1}{\sigma_{1u}} \right)^2 + \left( \frac{\sigma_2}{\sigma_{2u}} \right)^2 + \left( \frac{\tau_{12}}{\tau_{12u}} \right)^2 - \left( \frac{\sigma_{2u}}{\sigma_{1u}} \right) \left( \frac{\sigma_1}{\sigma_{1u}} \right) \left( \frac{\sigma_2}{\sigma_{2u}} \right) \quad (1)$$

Figures 7 through 10 are plots of DIST for typical net tension, shear-out, bearing, and combination failure modes respectively.<sup>1</sup> An initial application of the experimental failure load was used as the applied load for each computer run. The resulting contour plots were sufficient to predict the failure modes in all but the shear-out cases. For these specimens it was sometimes necessary to consider the ratios of lamina principal stresses to their respective ultimate stresses to differentiate between a plug type shear-out mode and a bending, tear-out mode.

<sup>1</sup>Figures 7a through 7d represent DIST contour plots of four laminae which compose a net tension failure specimen. A single plot of the major load carrying lamina for each of the other three failure modes is included to illustrate the contour patterns for these various modes.

Prediction of failure load was also made on the basis of the Hill criterion. The values of DIST in the first row of circumferential elements around the hole were considered for each lamina. A successive failure analysis similar to that discussed in [12] was used to predict ultimate load. As soon as an element in any given lamina achieved a value of DIST equal to 1.0 that lamina was assumed to have failed and was locally removed from the laminate. The load was then redistributed among the remaining laminae and all values of DIST were recalculated. If all recalculated values of DIST were less than 1.0 more load was applied until another lamina reached failure. This process was repeated until total laminate failure occurred.

The predictions of failure load based on equation (1) were always conservative. The degree of conservatism varied with failure mode, but more importantly it appeared to be a function of specimen anisotropy. To date only  $0^\circ/90^\circ/\pm 45^\circ$  specimens have been considered. The predicted failure loads for the net tension specimens improve greatly as the percentage of  $\pm 45^\circ$  lamina decreases (See Table 1). For example, for a 100% ( $\pm 45^\circ$ ) laminate the predicted failure load is about one-half the experimental failure load. For a ( $\pm 45^\circ/90^\circ$ ) laminate which contains 62.5% ( $\pm 45^\circ$ ) lamina the predicted failure load is about nine-tenths the experimental failure load. This same type of behavior was reported by Grumman Aerospace [13] in a study they performed on laminate tension data.

The Hill criterion was the only criterion of the three which was conservative in predicting failure load for each specimen investigated. Both the maximum strain failure criterion and the maximum stress failure

criterion overpredicted at least one specimen ultimate load. That is, even when the experimentally determined failure load was applied the ratio of principal strains (or stresses) to their respective ultimate strains (or stresses) did not exceed 1.0 as is required by these two criterion respectively.

Investigation of experimentally failed specimens exhibit excellent agreement with predicted failure behavior. For example, specimens which failed according to a slug type shear-out mode exhibited relatively smooth, clear fracture surfaces. The high values of DIST for the shear-out failure mode pictured in Figure 8 are a result of very high principal shear stress ratios in these regions, which would lend to rather smooth shear fracture surfaces. On the other hand, specimens which failed by a bending, tear-out failure mode (which is also considered a shear-out failure mode by some investigators) exhibited a very coarse, jagged fracture surface. This behavior is again expected from the computed stress ratios. Along lines at  $\pm 45^\circ$  in a  $(0^\circ/90^\circ/\pm 45^\circ)$  specimen, where the values of DIST are high, the largest stress ratios act in the first principal direction. These are the stresses which are trying to break fibers in tension. As a result, as the triangular section is being torn away from the specimen, fibers along these failure lines at  $\pm 45^\circ$  are being broken in tension; resulting in a very coarse, jagged fracture surface.

Another interesting feature of the experimentally failed specimens was the presence of a highly localized region of laminate destruction at the bolt-specimen interface. It was observed that a bearing failure of variable magnitude had occurred in conjunction with almost every other

type of experimental failure mode. This behavior was again predictable as is shown in Figures 7c, 7d, 8, 9 and 10.

#### 3.1.4 *Future Work*

Three important areas in this analysis where simplifications have been made will be investigated in the future.

- 1) The effects of interlaminar shear on the stress field near the hole.
- 2) The significant variation in material properties and ultimate allowables reported in the literature.
- 3) The non-linear stress-strain response of the composite materials.

The need for reliable off-axis failure data is also critical to the complete understanding of the failure of a composite structure under complicated loading. It is felt that continued investigation of the simple bolt bearing problem will yield further clues as to the mechanisms of failure due to stress concentrations.

It was also felt at the completion of this project that a similar failure analysis could be performed on more complex mechanically fastened joints made of composite materials. Such an investigation has been performed by this investigator and is reported in the next section.

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- [2] S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, Second Edition, McGraw-Hill Book Company, 80 (1951).
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- [12] Investigation of Joints in Advanced Fibrous Composites for Aircraft Structures, Air Force Flight Dynamics Laboratory, 13, (June 1969).
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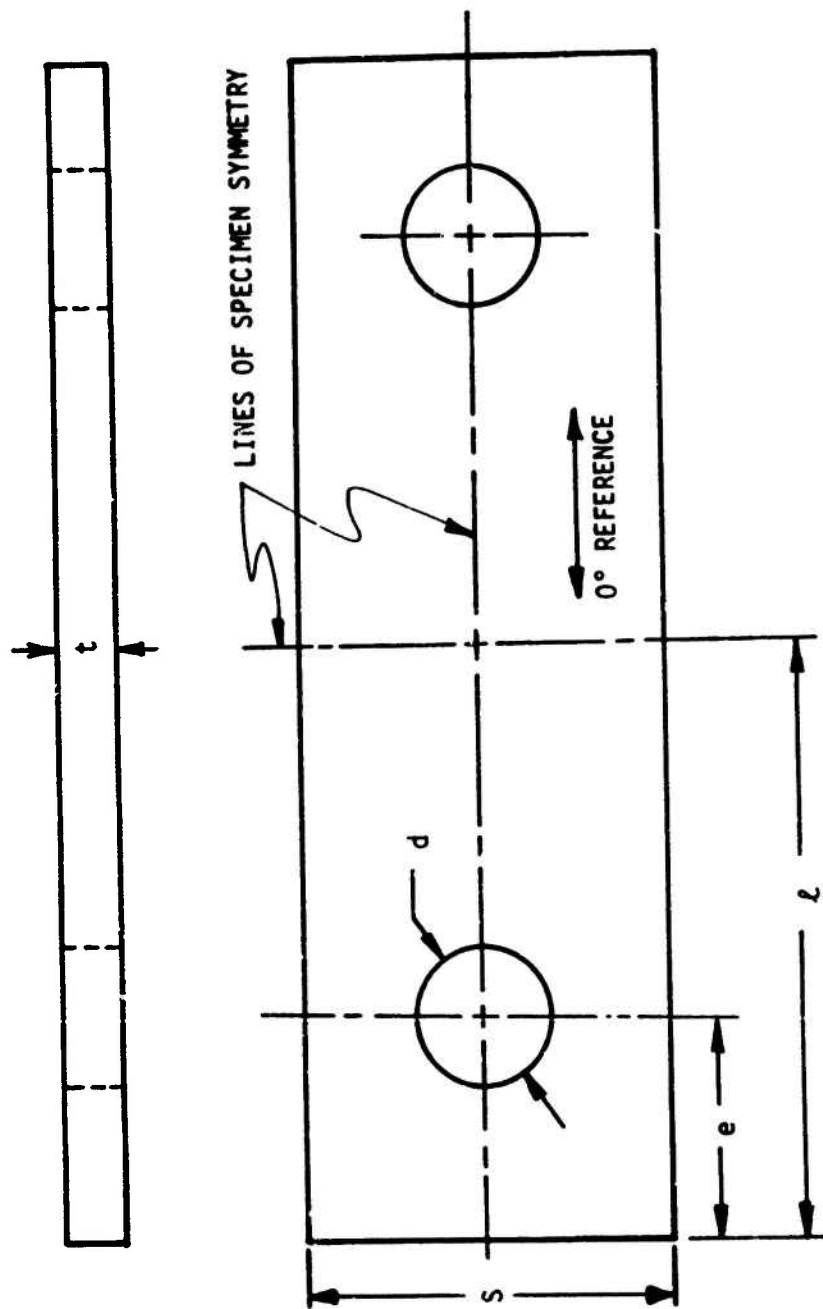
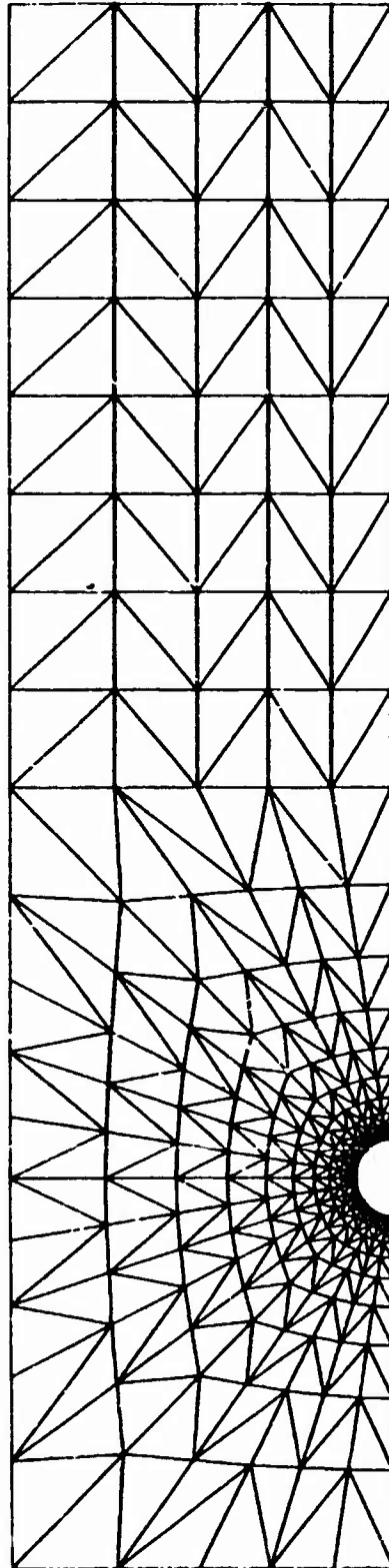


FIGURE 1: BOLT BEARING TEST SPECIMEN



480 ELEMENTS

279 NODES

FIGURE 2: BOLT BEARING GRID REPRESENTATION FOR  $E/D=5.0$ ,  $S/D=10.0$  &  $L/D=20.0$

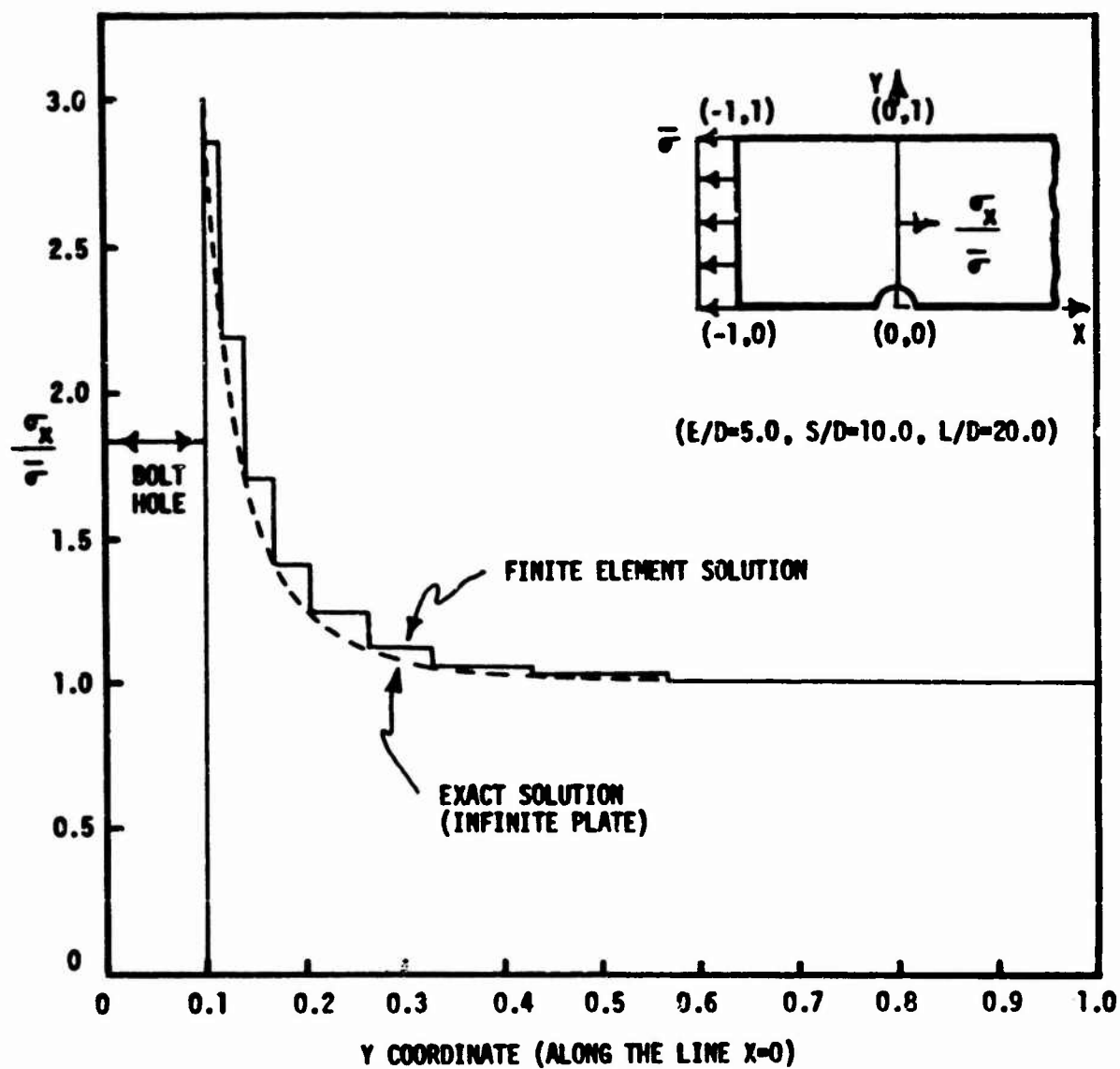


FIGURE 3a: ISOTROPIC BOLT BEARING VERIFICATION:  $\sigma_x / \sigma$  vs  $y$



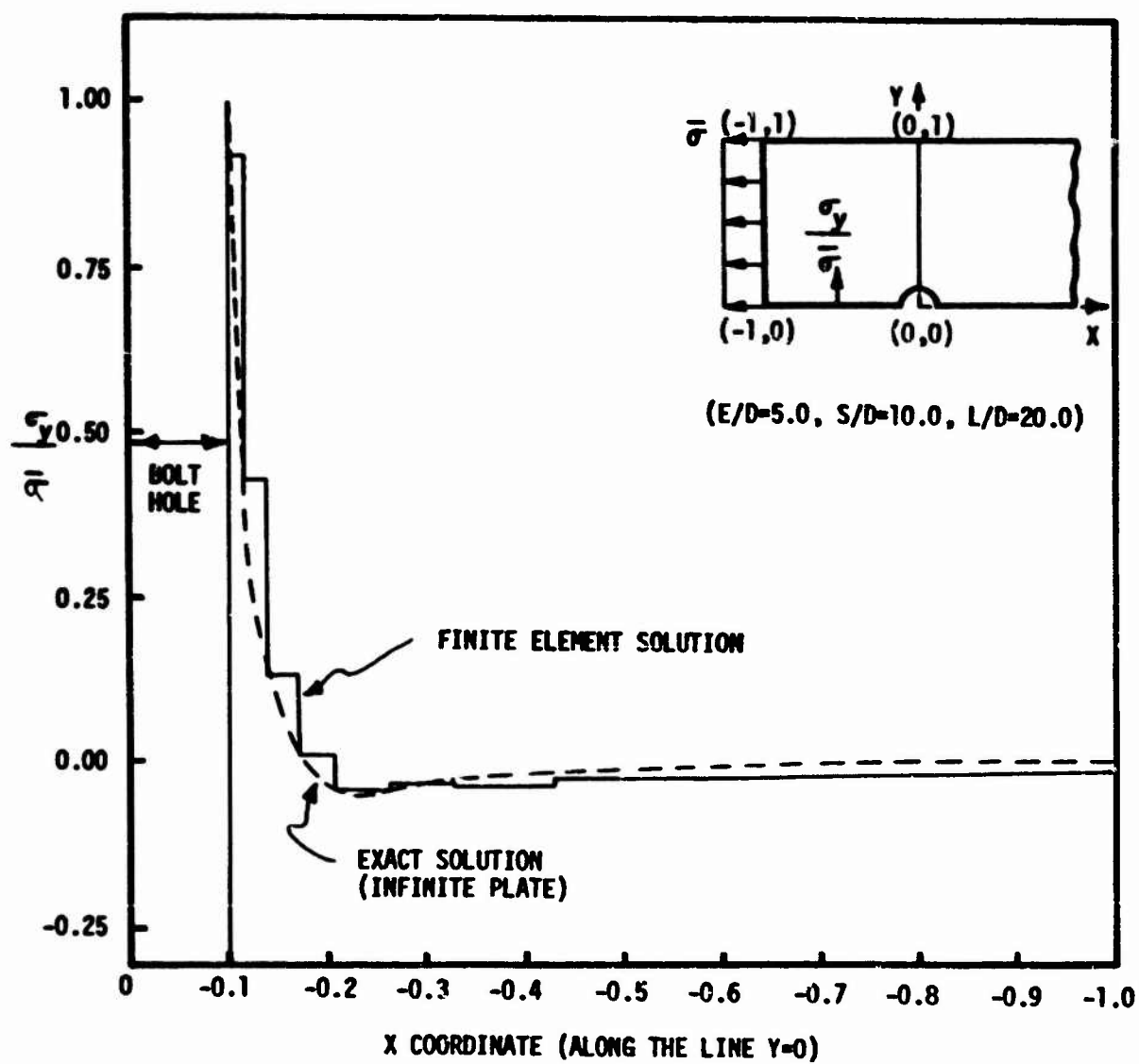


FIGURE 3b: ISOTROPIC BOLT BEARING VERIFICATION:  $\sigma_y/\sqrt{a}$  vs X

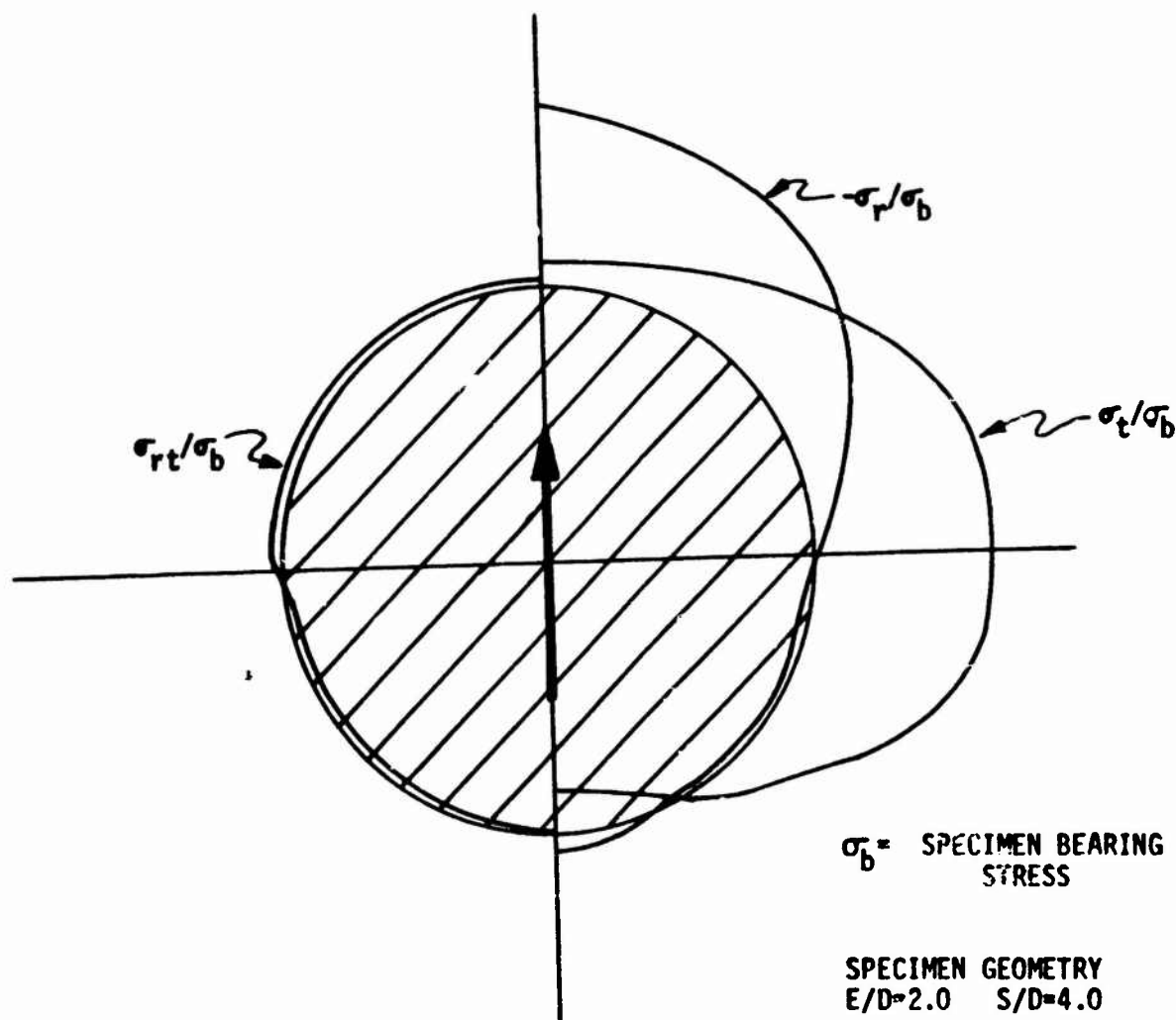


FIGURE 4: POLAR PLOT OF EDGE STRESSES FOR AN ISOTROPIC TEST RUN

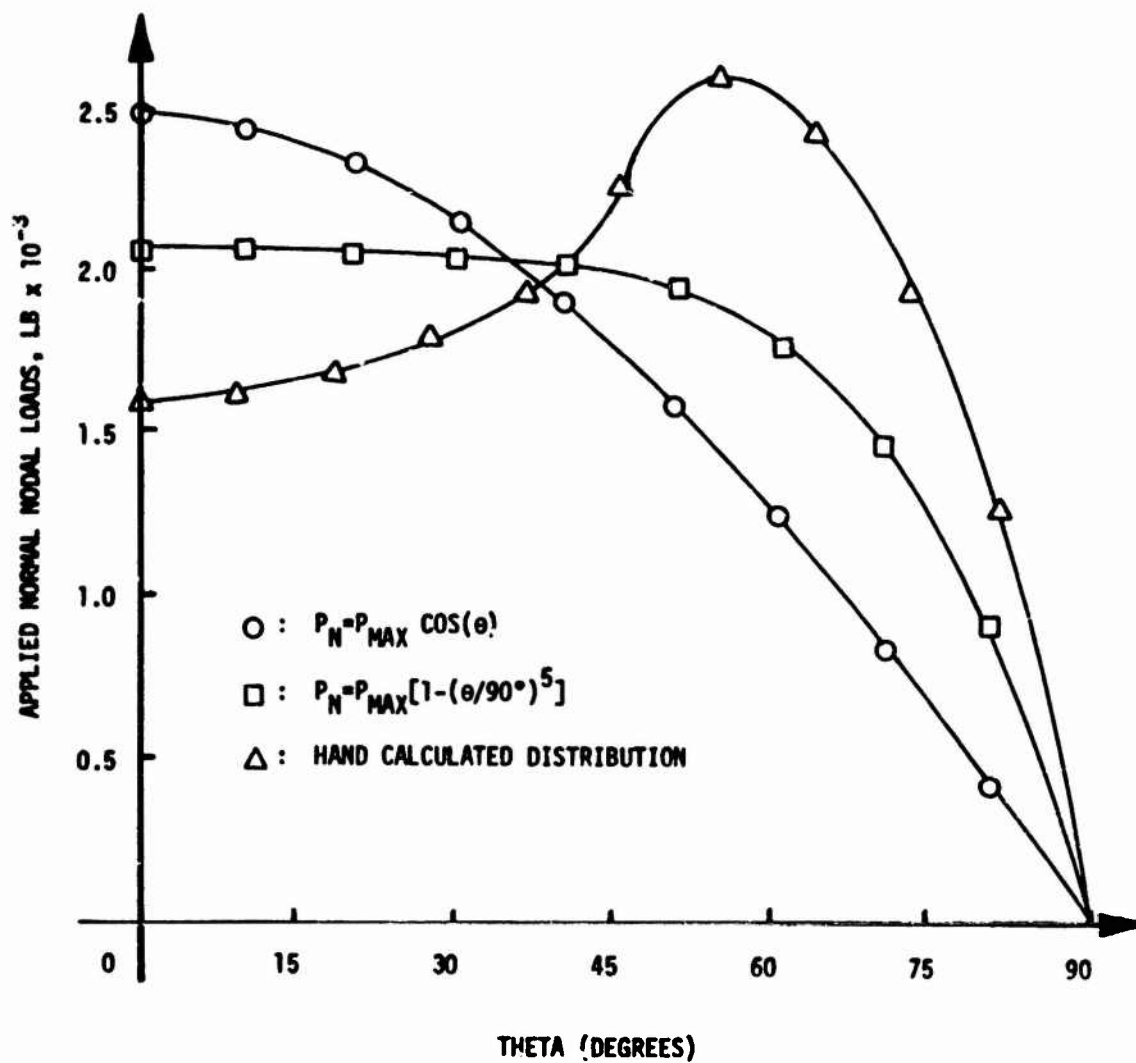
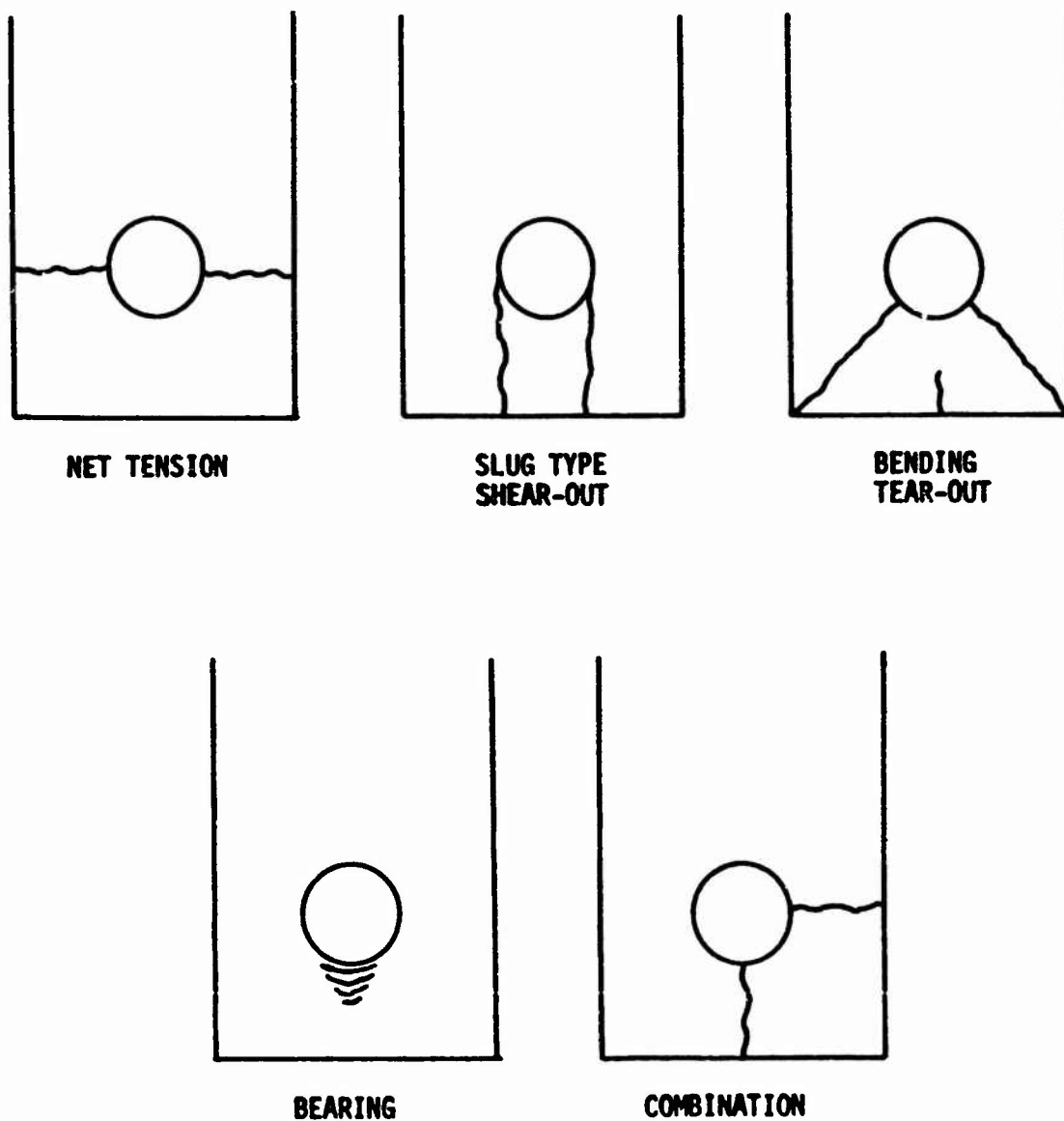


FIGURE 5: VARIATIONS ABOUT THE COSINE DISTRIBUTION OF NORMAL STRESS



**FIGURE 6: BOLT BEARING TEST SPECIMEN FAILURE MODES**



$E/D=2.0$ ,  $S/D=4.0$

$[(\pm 45/0)_5/90]_s$

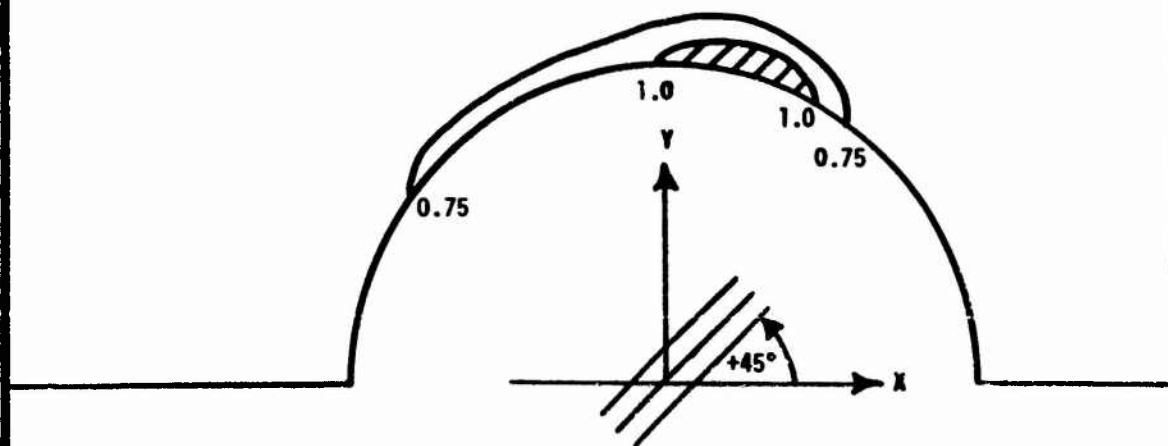
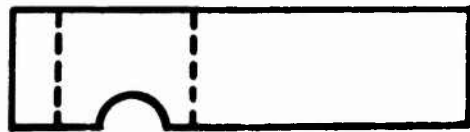


FIGURE 7a: NET TENSION FAILURE +45° LAMINA



$[(\pm 45/0)_s/90]_s$

$E/D=2.0, S/D=4.0$

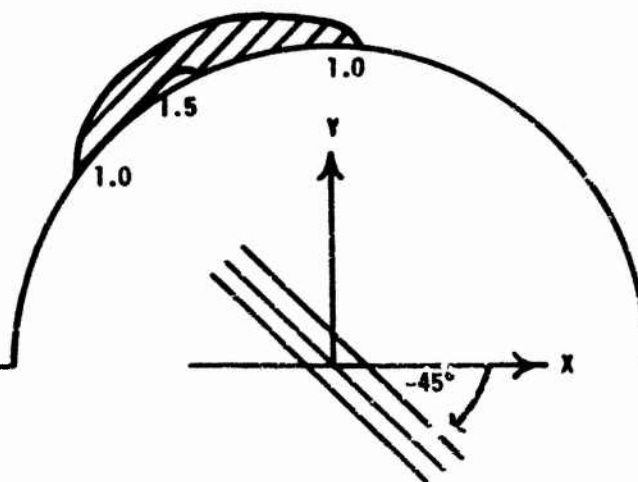


FIGURE 7b: NET TENSION FAILURE  $-45^\circ$  LAMINA



$[(\pm 45/0)_s/90]_s$

$E/D=2.0, S/D=4.0$

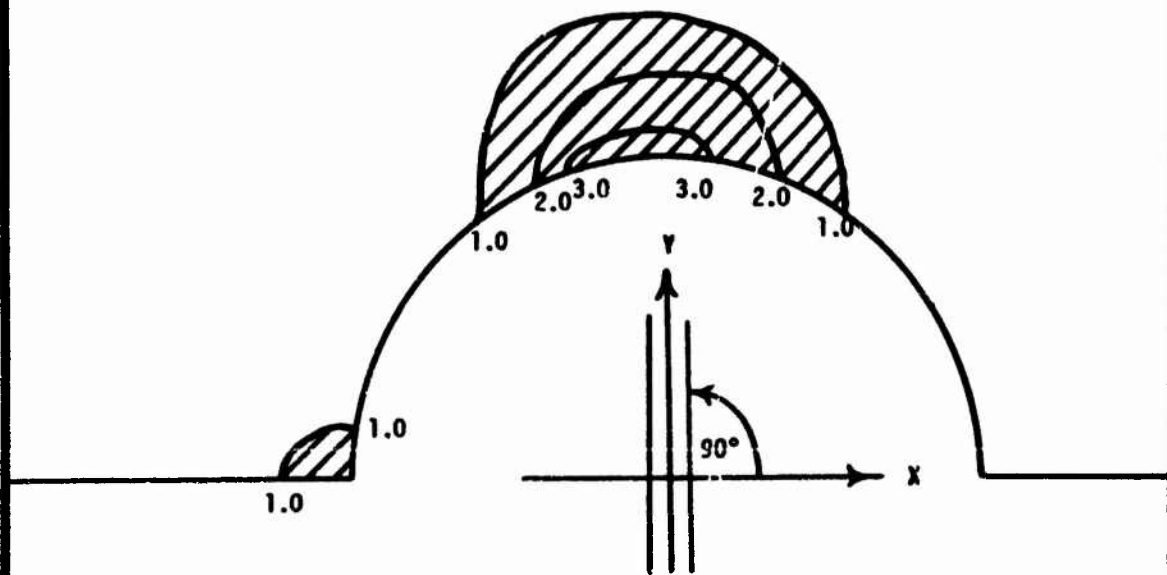
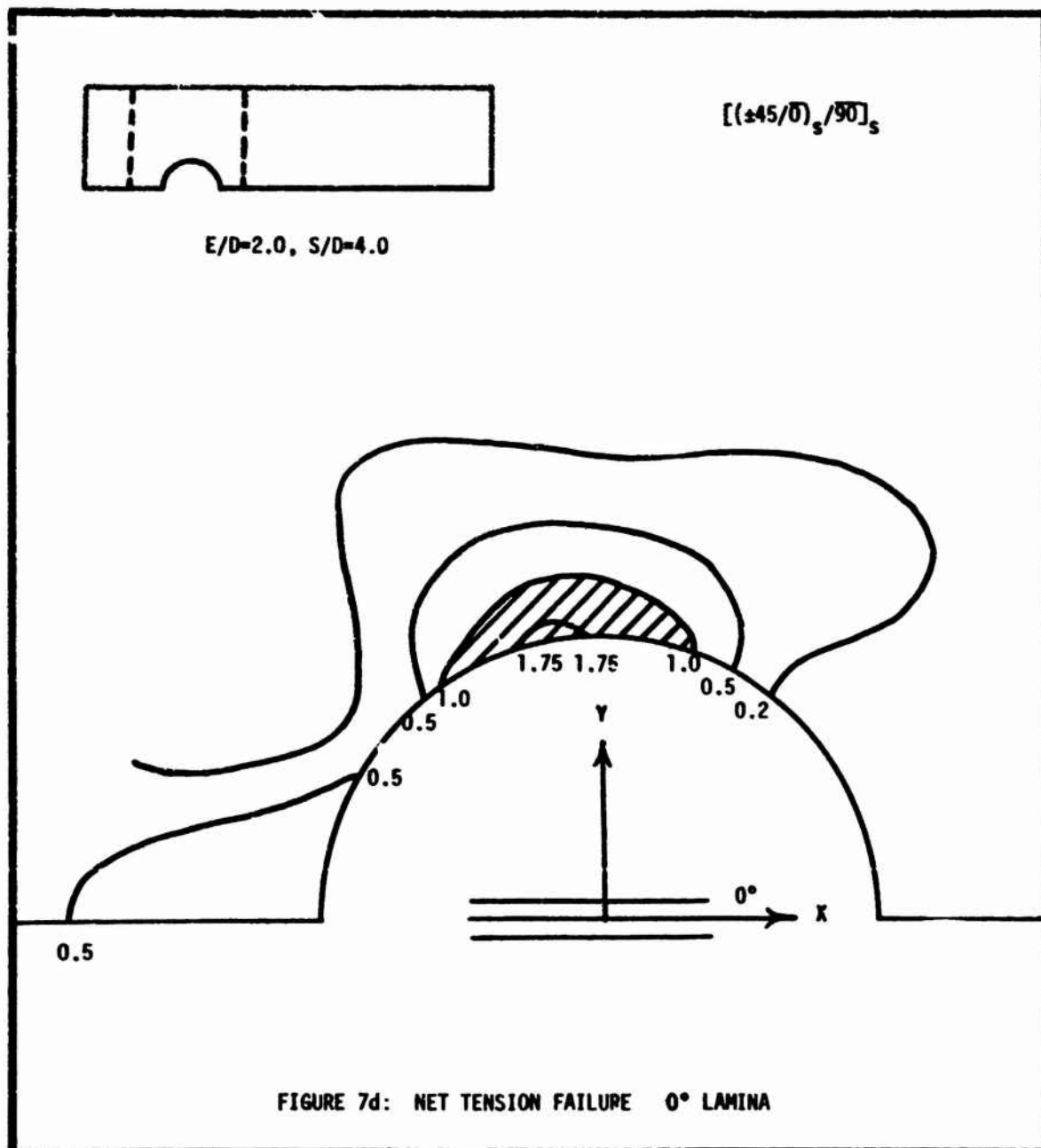


FIGURE 7c: NET TENSION FAILURE 90° LAMINA







$[\pm 45 / (0_6 / 90)_s]_s$

$E/D=6.0, S/D=6.0$

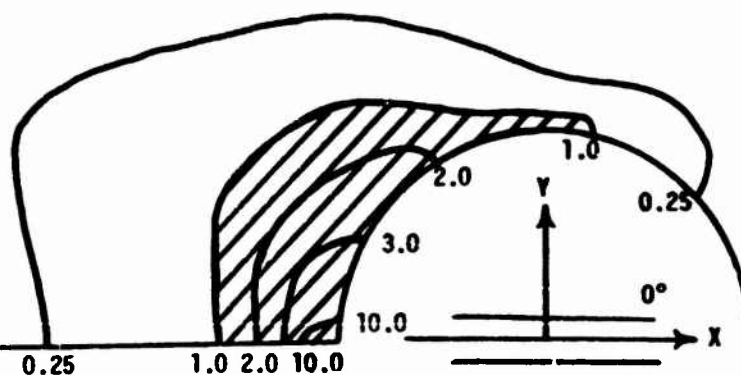
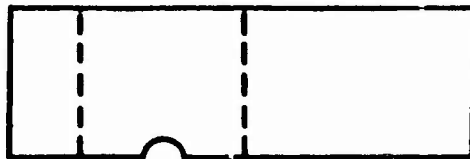


FIGURE 8: SLUG SHEAR-OUT FAILURE 0° LAMINA



$(0_5/\pm 45_5)$

$E/D=4.0$ ,  $S/D=7.53$

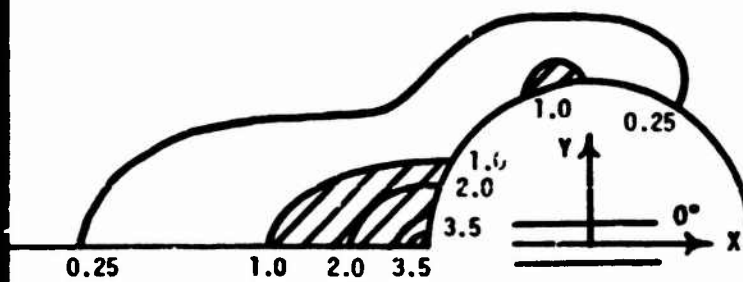


FIGURE 9: BEARING FAILURE  $0^\circ$  LAMINA



( $\pm 45^\circ$ )

$E/D=2.0$ ,  $S/D=4.0$

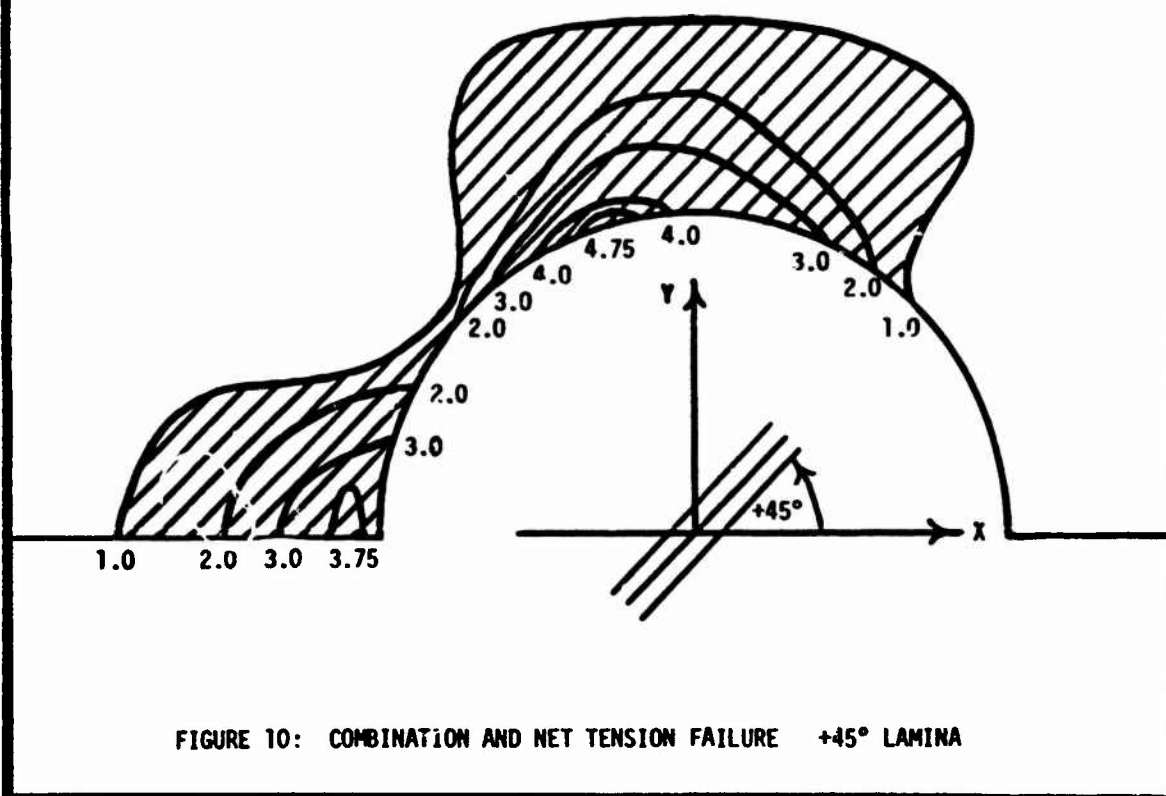


TABLE 1  
SUMMARY OF BOLT BEARING SPECIMEN DATA

Failure Mode	Laminate	Material	% $\pm 45^\circ$	% $0^\circ$	% $90^\circ$	$\frac{e}{d} : \frac{s}{d}$	Predicted Failure Load		
							DIST	Maximum Stress	Maximum Strain
T, T, C	$(\pm 45)$ $[(\pm 45/\bar{0})_s/\bar{90}]_s$ $(\pm 45_5/\bar{90}_6)$	G/E	100	-0-	-0-	2:4	0.47	0.47	0.62
T		B/E	72.7	18.2	9.1	2:4	0.76	0.58	0.49
T		B/E	62.5	-0-	37.5	3:4	0.93	0.94	1.00
S	$[\pm 45/(0_6/\bar{90})_s]_s$ $(0_6/\pm 45_5)$	B/E	13.3	80.0	6.7	6:6	0.58	0.61	1.89
S		B/E	62.5	37.5	-0-	2.5:7.53	0.98	1.10	0.95
B	$(0_6/\pm 45_5)$	B/E	62.5	37.5	-0-	4:7.53	~0.65	0.58	0.49

Nomenclature:    T    Tension    G/E    Graphite-Epoxy  
                          C    Combination    B/E    Boron-Epoxy  
                          S    Shear-Out  
                          B    Bearing

### 3.2 TOWARD A DESIGN PROCEDURE FOR MECHANICALLY FASTENED JOINTS MADE OF COMPOSITE MATERIALS

#### 3.2.1 Introduction

Currently much emphasis is being directed toward replacing metal components in weight sensitive structures, such as aircraft, with composite materials, due to their superior specific strength and specific stiffness properties. The potential weight savings which could result from such practices, however, have not, as yet, been realized.

Significant weight savings can be achieved throughout the bulk of a replacement component by tailoring the composite material to efficiently carry the loads which are known to occur in the existing metal component. The weight savings which result, however, are usually eliminated due to the inefficient joint<sup>2</sup> designs which are proposed by the designer to fasten the replacement component to the remainder of the existing structure. The measure of efficiency used here is simply load carried per pound of material. Thus, if a given load is to be carried by a structural member, the load carrying efficiency of that member increases as its weight is decreased.

In the design of metal joints only three failure modes need be considered; net tension, shear-out, and bearing. For a given metal the values of  $F_{TU}$ ,  $F_{SU}$ , and  $F_{BRU}$  can be experimentally determined and used to specify the joint parameters  $S$ ,  $E$ , and  $t$  respectively, given the bolt diameter. Thus, the design of metal joints is based on a very limited amount of experimental data.

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<sup>2</sup>The term joint will imply a mechanically fastened joint throughout the report.

Consider the complications which would arise if a similar empirical design procedure were used for joints made of composite materials. First of all, several additional failure modes are exhibited by composite joints which do not occur in metal joints, due to the anisotropy of composite materials. Thus, for a given laminate the amount of data required for design purposes would be about doubled. The major problem, however, is that the feasibility of obtaining effective stress allowables which can be related to geometric parameters for splitting, shear-out, or bending tear-out failure modes in composite materials has yet to be determined.

Secondly, consider the problems associated with the selection of joint lamination. The designer is using a material which may be tailored to satisfy certain design constraints which are application dependent. The number of possible lay up patterns which could be considered during a single design are innumerable. Thus, the amount of data acquisition which would be necessary to support an empirical design procedure in composite joints is prohibitive.

As a result, the designer is presently forced to select a laminate for which some data does exist. Since laminate effective stresses for the various failure modes are unknown, an overly conservative design must be proposed by the designer based on his interpretation of the available data. An overly conservative design, unfortunately, implies that additional material has been used wherever

necessary to compensate for a lack of confidence in predicting various failure modes. Such practices, of course, lead to inefficient designs.

There is one other important difference between metal joints and composite joints which should be mentioned. It can be deduced, using the results from [1], that the stress concentration factor which results in an anisotropic joint is greater than that which occurs in a geometrically similar isotropic joint. This is, of course, a disadvantage associated with using composite materials in joints. It is, however, more than compensated for by the materials specific strength and specific stiffness properties.

To recover the potential weight savings of designing with composite materials new design procedures must be proposed which will result in optimum joint designs with respect to total joint weight. It is the purpose of the reported study to investigate such improved design procedures.

A first attempt at such design procedures is proposed and is discussed in detail in Section 3.2.6. The procedures are sufficiently general that they may be used in conjunction with most available optimization routines. The results are being programmed by this investigator using an in-house pattern search optimization routine. Given valid input data, the program is designed to output that joint design in design space which has the minimum total joint weight while satisfying all the imposed design constraints. The results will, of course, only be as accurate as the assumptions on which the analysis

is based. As a result, further investigations regarding the accuracy of these assumptions is warranted and will be performed by this investigator.

As mentioned above, three failure modes have been observed in metal joints: net tensile shear-out, and bearing. Each of these modes exhibits ductile fracture behavior. In composite joints not only are there additional modes of failure to consider but fracture behavior ranges from ductile to brittle, depending on the failure mode being considered.

Finite element stress analyses of bearing and shear-out failures in composite materials [2] have shown that large regions of laminate destruction, on the order of a hole diameter in size, occur prior to actual laminate failure. It was also found that highly localized regions of laminate failure, about two orders of magnitude smaller than those required for bearing and shear-out failures, were present when net tension failures occurred. It is apparent, therefore, that these various failure mechanisms must be understood before a truly optimum joint design can be achieved, since a single failure criterion is not applicable to all the possible modes of failure in composite materials.

A recent study [3] has postulated the existence of a small but finite region of intense energy which supposedly governs failure in composite tension coupons. If stress concentrations induce such regions of intense energy in composite tension coupons a similar



phenomenon should occur in composite bolt bearing specimens. The results reported in [2], therefore, tend to support such a theory.

To further understand the phenomenon, a finite element study was performed for several composite tension coupons and is reported in Section 3.2.2. Again, very small, highly localized regions of laminate destruction were observed prior to actual failure. As a logical extension to the tension coupon experimental study [3] four geometrically similar bolt bearing specimens were designed, Section 3.2.2, to fail in net tension using a quasi-isotropic boron-epoxy material. These specimens are presently being fabricated and will be tested at General Dynamics, Fort Worth. If a characteristic crack length hypothesis is indeed valid, significant differences in applied failure stresses for these specimens should be observed. These differences should be predictable from the theory presented in [3].

The design procedures outlined and discussed in Section 3.2.6 are only intended to represent an initial attempt at moving toward the desired design procedures for joints made from composite materials. In Section 3.2.7 those areas which require further investigation are indicated.

### *3.2.2 Investigation of the Characteristic Crack Length Hypothesis*

Past experience with predicting net tension failure in anisotropic bolt bearing specimens using the distortional energy

failure criterion<sup>3</sup> [2] has shown that a small but finite region of material at the hole surface is always "past the point of failure" before laminate failure occurs. These regions were originally considered to result from an inherent conservatism of the finite element solution technique. A recent study performed at General Dynamics [3] has postulated the existence of a region of intense energy in composite tension coupons which seems to govern failure. The finite element results [2] in retrospect appear to support such a theory.

In the study performed at GD a series of graphite/epoxy tension coupons were designed and tested to failure. The specimens were identical in overall size and lamination but the sizes of the circular cutouts varied. If a similar series of metal specimens were tested it would be possible to predict the failure loads of all the specimens from the experimental failure load of a single specimen, using scaling factors which are only geometry dependent. In the case of the graphite/epoxy coupons a simple scaling of failure loads was not possible. It was found, however, that the observed failure behavior could be explained via fracture mechanics if the existence of a region of intense energy or a characteristic crack length was hypothesized. For a given laminate the size of the region was assumed constant.

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<sup>3</sup> It is well known that the Hill failure criterion is not a distortional stress energy. However, because of the close similarity with the isotropic failure criterion of distortional energy, the phrase "distortional energy failure criterion" will denote the Hill failure criterion as used in Section 3.1.

As previously mentioned highly localized regions of predicted laminate failure have been observed via finite elements in composite bolt bearing specimens. An investigation to determine whether or not similar regions could be observed via finite elements in composite tension coupons has since been completed. Two specimens were selected from [3] for analysis. The failure loads predicted by the theory [3] for these two specimens were used as applied loads for the computer runs. Using the most recent graphite/epoxy material constants and ultimate allowables it was found that for a tension coupon containing a 1.0 inch diameter hole the region of localized failure measured 50 mils. Likewise, for a specimen containing a 0.2 inch diameter hole the region measured 31 mils. The characteristic crack length proposed for the laminate used in the actual specimens was approximately 40 mils and agrees quite well with the finite element results. Distortional energy contour plots for the various laminae in the 1.0 inch diameter specimen are shown in Figure 1. Localized lamina failure is predicted to occur when the value of the normalized distortional energy exceeds 1.0 [2].

There are several reasons to suspect that the values of the distortional energies near the hole are not entirely accurate. Finite element size differences in the two specimens at the hole surface, the effects of interlaminar shear at the circular boundaries, and uncertainties regarding the cross term in the distortional energy

failure criterion probably account for a large percentage of any possible error. The fact that a small region of material appears to be "past the point of failure" in both specimens before laminate failure occurs, however, is the significant result rather than the actual sizes of these regions.

If a region of intense energy actually governs failure in composite tension coupons it should also govern failure in bolt bearing specimens made of the same material. Thus, four geometrically similar bolt bearing specimens were sized using a quasi-isotropic graphite/epoxy laminate to see if differences in experimental failure loads could be observed and explained using the characteristic crack length hypothesis. The equations presented in [4] were used to size the initial design, Table 1. The ultimate load predicted by the equations for a net tension failure was slightly less than that necessary for a bearing failure and only about two thirds that necessary for a shear-out failure. A computer analysis of the proposed specimen configuration indicated that a net tension failure would occur at precisely the load predicted by the equations. The resulting distortional energy plots for the initial design are shown in Figure 2a.

Note, however, that the results reported in [2] indicate that before a bearing failure may occur in a bolt bearing specimen a large region of material directly ahead of the bolt must exhibit normalized distortional energies greater than 1.0. Thus, the computer

analysis predicts a net tension failure to occur well ahead of any possible bearing failures, Figure 2a. This, however, disagrees with the behavior predicted by the equations.

At the request of Mr. J. R. Eisenmann the specimen was resized to eliminate even the remotest possibility of premature bearing failures since such failures would give no information regarding the possible presence of a characteristic crack length. In the revised design (Table 2) the specimen width has been decreased and the edge distance increased. The equations now predict a net tension failure to occur well ahead of both bearing and shear-out failures. A computer analysis of the revised design again indicated that a net tension failure would occur (See Figure 2b). The failure load predicted by the computer analysis, however, was 58% greater than the failure load predicted by the equations. Both the equations and the finite element analysis agree that a net tension failure will occur well ahead of both bearing and shear-out failures. The two methods disagree significantly, however, on the predicted failure loads.

These differences in predicted failure loads indicate clearly that basing bolt bearing specimen failure predictions on the equations presented in [4] is very dangerous. The equations are empirical in origin and only apply to a limited range of specimen geometries. The revised design is obviously outside the region of applicability.

### *3.2.3 Review of Past Design Programs Involving Composite Joints*

Two programs involving the design and testing of joints made of composite materials were recently completed at General Dynamics, Fort Worth. In the original program [5] specimens were sized to fail in net tension at the innermost row of bolts. The maximum load to be carried by a joint was first specified. An estimate as to bolt load partitioning was next made based on the designers understanding of load distributions in isotropic joints. The laminate to be used was selected and the joint dimensions were then scaled from existing single- and double-fastener coupon data.

During testing, eight of the nine specimen designs failed in a splitting mode rather than the desired net tension mode. Thus the techniques used in sizing these joints proved to be unsatisfactory.

In the second joints program [6] only one joint was designed and tested. The maximum load to be carried by the joint was again specified. The designer assumed that each bolt in the joint would carry an equal percentage of the total joint load at failure. The longitudinal strains in the splice plate and main plate were set equal at two locations in the joint; midway between the first two rows and last two rows of bolts. The specimen was sized at these two locations to fail in net tension at the innermost row of bolts. A linear taper in both geometry and lamina thicknesses was then employed. The resulting joint design was built and tested. It failed in net tension at the innermost row of bolts as desired.

The major criticism of the latter design procedure is that it ignores the interaction between bolt load partitioning and joint geometry. The design procedures proposed by this investigator include such interaction relationships. The following section, Section 3.2.4 describes the proposed load partitioning analysis in detail. In Section 3.2.5 the analysis technique is used to predict bolt load distributions for six specimens selected from [5] and [6]. The results are used as input data for finite element analyses of the various specimens. A joint failure criterion is then proposed which, when applied to the finite element results, successfully predicts failure modes and conservatively predicts failure loads for each of the specimens.

### 3.2.4 Evaluation of Load Partitioning in Joints

To design a joint one must first understand the way in which changes in joint geometry affect bolt load partitioning. Two methods for predicting bolt load distributions in a given joint are proposed here. The first will be referred to as the point strain matching technique, and the second, as the displacement matching technique.

In both techniques only a single column of bolts will be considered. Larger joints may be constructed from identical columns of bolts connected to one another along their common sides. When stress analyses are performed for such joints curves presented in [7] will be used to correct for the effects induced by the adjacent columns. Both techniques assume that all bolts act as rigid pins and that the effects of plate bending are negligible.

#### 3.2.4.1 Point Strain Matching Technique

In the point strain matching technique the average longitudinal strain in the main plate,  $\epsilon_m$ , is equated to the average longitudinal strain in the splice plates,  $\epsilon_s$ , midway between each set of adjacent bolts in a given column. Referring to Figure 3 we may write

$$\epsilon_m (i, i + 1) = (F - \sum_{k=1}^i P_k) / (E_{xm} A_m) \quad (1)$$

and

$$\epsilon_s (i, i + 1) = \frac{1}{2} \sum_{k=1}^i P_k / (E_{xs} A_s) \quad (2)$$



The notation  $(i, i + 1)$  implies evaluation at the midpoint between the bolts labeled  $i$  and  $(i + 1)$ . Note that equations (1) and (2) are written for joints loaded in double shear. The equations may be used, however, for joints which are loaded in single shear if one half the total cross sectional area of the single shear splice plate at the various midpoints is substituted for  $A_s$ .

The assumption is now made that

$$\epsilon_m(i, i + 1) = \epsilon_s(i, i + 1) \quad (3)$$

Substituting equations (1) and (2) into (3) and rearranging we have:

$$\sum_{k=1}^i P_k = \frac{F}{1 + \frac{1}{2} \left( \frac{E_m(x) A_m(x)}{E_s(x) A_s(x)} \right)}_{i, i + 1} \quad (4)$$

Equation (4) may be evaluated for  $i = 1, N - 1$ , where  $N$  is the total number of bolts per column. Thus, equation (4) represents a total of  $(N - 1)$  equations in  $N$  unknowns, namely  $P_1$  thru  $P_N$ . One other equation can be written which relates the individual bolt loads. It is, of course, the overall joint equilibrium equation.

$$F = \sum_{k=1}^N P_k \quad (5)$$

For a given specimen the modulus and cross sectional area of both the main plate and splice plates are known at every point along the specimen. Therefore, equations (4) and (5) can be used to solve directly for  $P_1$  thru  $P_N$ .

### 3.2.4.2 Displacement Matching Technique

In the displacement matching technique the change in length of a section of main plate between two adjacent bolts is equated to the change in length of the section of splice plates between the same two bolts. That is

$$\Delta \ell_m (i \rightarrow i+1) = \Delta \ell_s (i \rightarrow i+1) \quad (6)$$

Equations (1) and (2) may be rewritten as follows:

$$d\ell_m = \frac{(F - \sum_{k=1}^i P_k) dx}{E_m(x) A_m(x)} \quad (7)$$

$$d\ell_s = \frac{(\frac{1}{2} \sum_{k=1}^i P_k) dx}{E_s(x) A_s(x)} \quad (8)$$

These equations require that the modulus and cross sectional area of both the main plate and splice plates be expressed as functions of  $x$ . Integrating (7) and (8) with respect to  $x$  from  $x_i$  to  $x_{i+1}$  and substituting into (6) it follows that

$$\sum_{k=1}^i P_k = -\frac{M}{M + S/2} \times F \quad (9)$$

where

$$M = \int_{x_i}^{x_{i+1}} \frac{dx}{E_m(x) A_m(x)} \quad \text{and} \quad S = \int_{x_i}^{x_{i+1}} \frac{dx}{E_s(x) A_s(x)} \quad (10)$$

As before from equilibrium we have

$$F = \sum_{k=1}^N P_k \quad (11)$$

Equations (9) and (11) represent  $N$  equations in  $N$  unknowns which may be solved directly for  $P_1$  thru  $P_N$ .

The point strain matching technique is used in Section 3.2.5 to calculate load distributions for the specimens analyzed. The load distribution for one of the specimens was calculated a second time using the displacement matching technique. A comparison of the results is shown in Section 3.2.5. The differences in load distributions are seen to be negligible.

The displacement matching technique is based on a more realistic assumption regarding physical joint behavior than is the point strain matching technique. Thus the reader may prefer to use the displacement matching equations in the proposed joint synthesis procedure discussed in Section 3.2.6. Further investigation regarding possible differences in the predicted behavior of the two techniques is warranted.

### 3.2.5 *Computer Analysis of Experimentally Failed Composite Joints*

The purpose of the analysis phase was to establish a proposed joint failure criterion which could be automated and included in the final optimization program. The proposed criterion should be able to predict both joint failure location and failure mode. It should also be conservative in predicting failure loads and as simple operationally as possible.

Six joints designed and tested at General Dynamics were selected from [5] and [6] and were analyzed via finite elements. Table 3 describes these various joints in detail.

Analyzing a complete joint in a single finite element run with any degree of accuracy was impossible due to computer storage limitations. It was, in fact, only possible to analyze one hole at a time to achieve suitable accuracy.

Thus, the following analysis procedure was used. Each of the joints analyzed consisted of a number of identical columns of bolts as illustrated in Figure 4a. It was assumed that each column could be analyzed separately and that each carried an equal share of the total joint load which was present at failure. The joint geometries of six specimens selected for investigation were such that if the joints were made of an isotropic material the effects of adjacent columns of bolts would be negligible [7]. The equations from [1] indicate that the stress concentration factors which result

in anisotropic tension coupons are always greater than the stress concentration factors which result in geometrically similar isotropic tension coupons. It is reasonable to assume that the same holds true for bolt bearing specimens. Thus the assumption was made that the effects of adjacent columns of bolts were negligible in the actual composite joints since for the same applied loads a greater stress concentration factor implies a more rapid stress field decay.

To determine the effects of adjacent columns of bolts on the column of interest in the synthesis routine the graphical results from [7] will be used due to a lack of similar information for composite materials. Thus, conservative designs with respect to specimen width will result. Excessive conservatism implies a wasting of material and unwanted weight. Thus the degree of conservatism which results from using the correction factors from [7] will be investigated in the future.

The point strain matching technique was used to determine the bolt load distribution for each of the six joints. The resulting distributions are shown in Figure 5. The displacement matching technique was only applied to one specimen, specimen 6, for reasons of comparison with the point strain matching technique. The displacement matching results are included in Figure 5 and are denoted by the dashed lines. The differences between the two sets of results are seen to be negligible.

As mentioned above it was necessary to isolate single bolt holes for analysis to achieve suitable finite element accuracy. The holes which were selected for analysis were modeled as single fastener coupons as shown in Figure 4b. Each hole in specimen 6 was analyzed while only the first and last holes were analyzed for specimens 1 thru 5.

The stress boundary conditions for the resulting single fastener coupons were determined from the bolt load distribution results in the following manner. Consider the  $i^{th}$  hole in the column of bolts illustrated in Figure 4c. The load carried by the  $i^{th}$  bolt is  $P_{Bi}$ . From equilibrium considerations we require that a skin load,  $P_{Si}$ , of magnitude

$$P_{Si} = \sum_{k=i+1}^N P_{Bk} \quad (12)$$

be carried by the leading edge of the  $i^{th}$  coupon. The skin load was applied to the leading edge of the coupon as a uniform stress in the computer analyses. In the actual specimens, however, the material surrounding a given bolt hole does not see a uniform skin stress in the vicinity of the preceding loaded hole unless the holes are separated by a sufficient amount of material. Compare the actual stress distribution at the leading edge of the imaginary coupon, Figure 6a, with the uniform stress distribution imposed at that

boundary in the finite element analysis, Figure 6b. The amount of load which must flow around the bolt hole in Figure 6b is significantly greater than that in Figure 6a. Thus the resulting stress concentration factor in the computer analysis will be greater than that which occurs in the actual specimen.

Corrections were made to the computed stress concentrations using [8] in an attempt to account for the error induced through the use of the uniform skin stress boundary condition. Distortional energy contour plots for the six specimens analyzed are shown in Figures 7 thru 12. It has been found by this investigator [2] that such plots are extremely convenient for data presentation. In regions of high distortional energies the principal stress ratios which are dominant have been indicated. Table 4 summarizes the important information contained in these figures.

Figures 7 thru 11 are for the five specimens selected from the original testing program at General Dynamics [5]. The first four specimens failed experimentally in splitting modes which appear to originate, upon examination of the specimens, at the last row of bolts. The fifth specimen, Figure 11, failed experimentally in net tension at the first row of bolts. Figure 12 represents the single specimen tested in the second General Dynamics program [6]. It also failed in net tension at the first row of bolts. The experimental failure behavior of these specimens, in conjunction with the stress analysis results illustrated in the figures, was used in the development of the proposed joint failure

criterion. A description of how the criterion evolved is presented below.

Consider for the moment Figures 7 thru 10. The stress patterns are identical; the values only differ slightly. The discussion which follows for Figure 7 is also valid for Figures 8 thru 10. A region of very high distortional energies occurs in the  $0^\circ$  laminae directly ahead of the last row of bolts in the specimen. The  $\sigma_2/\sigma_{2ut}$  stress ratios are dominant in the region, which implies local matrix failure (splitting). The maximum value of  $\sigma_1/\sigma_{1uc}$  in the region is 0.49. Results from [2] indicate that once the  $0^\circ$  laminae split (i.e.,  $\sigma_2/\sigma_{2ut} > 1.0$ ) a value of  $\sigma_1/\sigma_{1uc} \geq 0.65$  is necessary to cause a bearing failure to occur. Thus, even though the  $0^\circ$  laminae have split, Figure 7, the values of  $\sigma_1/\sigma_{1uc}$  are not large enough to cause a bearing failure to occur.

It has been assumed here that matrix failure does not significantly degrade the laminate since the percentage of hoop load carried by the  $0^\circ$  laminae directly ahead of the bolt was small. However, in specimens where a large percentage of the hoop load is carried by the matrix prior to failure a similar assumption is not possible. Consider a specimen consisting of almost all  $0^\circ$  laminae and only a few  $\pm 45^\circ$  laminae. Matrix failures in the  $0^\circ$  laminae would result in significant load transfer from the  $0^\circ$  laminae to the  $\pm 45^\circ$  laminae. Even if laminate failure did not occur as a result of the load transfer the laminate would be significantly damaged. It is apparent, therefore,



that a successive failure analysis must be included in the final design procedure to account for such load redistribution.

High distortional energies also result in the  $0^\circ$  laminae at  $\theta = 90^\circ$ , Figure 7, at both the first and last row of bolts due to large values of  $\sigma_1/\sigma_{1ut}$  in these regions. The maximum distortional energy value at the last row of bolts, 2.0, is greater than the maximum value at the first row, 1.5. The same is true of the maximum values of  $\sigma_1/\sigma_{1ut}$  in these two regions. If the  $0^\circ$  laminae were to fail, the  $\pm 45^\circ$  laminae would not be able to carry the additional load transferred to them from the  $0^\circ$  laminae; as a result, laminate failure would occur. Thus a net tension failure at the last row of bolts is the most probable failure mode indicated from the results so far.

The assumption that matrix failure does not significantly degrade the laminate is valid throughout specimen 1. The regions of high distortional energies which result from large  $\sigma_2/\sigma_{2ut}$  ratios are therefore eliminated from consideration. The only remaining region of interest is the one in the  $+45^\circ$  laminae which occurs at the last row of bolts where the fibers are tangent to the hole. Both the maximum distortional energy value and maximum  $\sigma_1/\sigma_{1ut}$  value in this region are greater than the corresponding values which indicated a net tension failure at the same hole. Once the  $+45^\circ$  fibers break in tension the remainder of the laminate cannot carry the existing load and laminate failure also occurs. Thus a splitting mode is favored over the net tension mode previously indicated for

specimen 1. Since all the possible regions of failure initiation have been examined a splitting mode is predicted by the analysis. The actual specimen did indeed fail in a splitting mode. The predicted failure load is conservative. If  $P_F$  is the actual experimental failure load for specimen 1 the distortional energy failure criterion predicts failure to occur at  $P_F/\sqrt{2.5}$  or  $0.64 P_F$ . The maximum stress failure criterion predicts failure to occur at  $P_F/1.5$  or  $0.67 P_F$ . Analyses of specimens 2 thru 4 yield very similar results.

Following the same procedure it can be deduced that a bearing failure does not occur in specimen 5, Figure 11. Both the distortional energy and maximum stress failure criteria conservatively predict a net tension failure to occur at the first row of bolts, which again agrees with the experimental failure mode.

Predicting failure for specimen 6 is slightly more complicated. The finite element results, Figure 12, indicate that  $\sigma_1/\sigma_{1uc}$  reaches 0.65 at the last row of bolts in the  $0^\circ$  laminae prior to matrix failure,  $\sigma_2/\sigma_{2ut} = 1.0$ . A bearing failure is predicted to occur in such a case at that load where either  $\sigma_2/\sigma_{2ut}$  reaches 1.0 or  $\sigma_1/\sigma_{1uc}$  reaches 1.0, whichever occurs first. The assumption is made at bearing failure initiation that the bolt causing the bearing failure to occur is unable to carry any additional load during subsequent specimen loading. The additional applied load is distributed among the remaining bolts in the column in proportion to

the loads which they carried at bearing failure initiation. Figure 12 is the finite element representation of the state of stress present in specimen 6 when the experimental failure load was applied. Notice that  $\sigma_2/\sigma_{2_{ut}} = 1.0$  and  $\sigma_1/\sigma_{1_{uc}} = 0.80$  in the  $0^\circ$  laminae directly ahead of the last row of bolts. The material ahead of the last row of bolts has failed in bearing and load redistribution, as described above, has taken place. A load distribution plot for specimen 6 is illustrated in Figure 12. Note that the revised load distribution plot is much more uniform than that of Figure 5(f).

Observe that regions of high distortional energies do not occur in the vicinity  $\theta = +45^\circ$  at any of the bolt holes except at the last row of bolts in the  $-45^\circ$  laminae. Matrix failure is on the verge of occurring here. Net tension failures, however, are indicated at various locations along the specimen which would occur prior to matrix failure in the  $-45^\circ$  laminae. A splitting mode is, therefore, definitely not indicated by the distortional energy plots.

Regions of high distortional energies in the  $0^\circ$  laminae at  $\theta = 90^\circ$  are present at the first and third through sixth rows of bolts. The maximum distortional energy value, 1.2, occurs at the fifth row. The largest value of  $\sigma_1/\sigma_{1_{ut}}$  in these four regions of interest is 1.03, which also occurs at the fifth row. Thus, both the distortional energy and maximum stress failure criteria predict a net tension failure to occur in specimen 6 at the fifth

row of bolts at  $0.97 P_F$ , where  $P_F$  is the actual experimental failure load for the specimen. The predicted failure load is again conservative but only by about 3% as opposed to about 35% for specimens 1 thru 5. The predicted failure mode was again correct but the location was not.

The analysis of specimen 6 shows that the design was a good one, in that each hole was close to failure when the joint failed experimentally. Notice also that the various bolts were fairly equally loaded when joint failure occurred. Some designers feel that such a bolt load distribution is necessary if a joint is to carry load efficiently. The validity of such a statement can only be determined by further analytical and experimental investigation.

Thus a joint failure criterion has been proposed which has successfully satisfied the requirements imposed on it at the beginning of Section 3.2.5. The criterion was able to predict both failure location and failure mode in all but Specimen 6 where it incorrectly predicted failure location. More importantly it was able to conservatively predict failure loads for each specimen. It was found that the maximum stress failure criterion agreed with the distortional energy failure criterion to within just a few percent in predicting failure loads. It was also found that failure was always initiated at locations around hole

surfaces where fibers were being broken in tension and were tangent to the hole surface.

Therefore, to make the failure criterion as operationally simple as possible we need only check for fiber failures where the fibers run tangent to the hole surface. A successive failure analysis must be performed at these locations to insure against premature failures induced by matrix failures in other laminae. The successive failure analysis should also be performed in regions where fibers are perpendicular to the hole surface since matrix failures in these laminae may also induce premature laminate failures.

### 3.2.6 *Proposed Mechanically Fastened Joint Design Program*

The preceding sections explain the procedures one would go through if a given joint were to be analyzed. A method is now proposed by which a joint may be designed to meet certain design constraints while attempting to minimize total joint weight.

An outline of the proposed mechanically fastened joint synthesis program is presented below to give a general understanding of the procedures involved in arriving at an optimum joint design with respect to total joint weight. The various procedures are then discussed in detail.

#### 3.2.6.1 Outline of Proposed Synthesis Program

- (A) Specify the known input data.
- (B) Determine the design variables and their range of allowable values.
- (C) Specify the necessary design constraint equations which will insure that joint failure does not occur until the design ultimate load is reached. The design ultimate load will be included as part of the input data.
- (D) Specify an initial design.
- (E) Calculate the various bolt loads for the current proposed geometry using the bolt load partitioning results, Section 3.2.4.
- (F) Perform a stress analysis of the proposed design to determine the average laminate stresses at various critical points along each of the circular boundaries. A closed form solution to the problem

illustrated in Figure 13a, based on the theory presented in [9], will be used to perform the required stress analyses. Corrections will be made to account for the effects of finite specimen size from [10].

- (G) Transform these average laminate stresses to lamina stresses.
- (H) Determine whether any of the lamina stresses exceed the design constraints imposed in (C).
- (I) Assign penalty functions to the weight function for each design constraint which is not satisfied and calculate the total weight for the proposed joint design.
- (J) Select a new design by moving in design space along a path which tends to decrease the total weight function.
- (K) Repeat (E) thru (J) until a suitable optimum design is achieved.
- (L) If desired, a detailed stress analysis may be performed for the proposed optimum design using finite elements, Section 3.2.5. A final check may be necessary since the stress analyses performed in (F) are based on isotropic correction factors.

### 3.2.6.2 Discussion of Program Details

#### 3.2.6.2.1 Input Data

The following information will be read into computer program as input data. It may be desirable in later work to include one or more of these parameters as program variables. The diameters of the bolts used in a given joint will all be the same, D<sup>4</sup>. The actual size

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<sup>4</sup>Underlined symbols and phrases denote input information.

will be dictated by joint application as well as by standard size limitations. Given an effective shear allowable for the bolt material,  $F_{SU}^B$ , and the maximum load to be carried by the joint per column of fasteners,  $F$ , one may determine a safe number of bolts,  $N$ , to be used in a given column since it appears from the analysis of specimen 6 that a fairly uniform load distribution is desirable.

Selection of the splice plate material will be application dependent. The material will probably be either a high strength steel or titanium. In either case the values of  $F_{TU}^S$ ,  $F_{SU}^S$ , and  $F_{BRU}^S$  must be input so that constraint relationships may be later defined to insure against splice plate failures in net tension, shear-out and bearing respectively. In the load partitioning calculations the splice plate modulus,  $E_S$ , is also required.

Similarly a decision must be made as to whether boron/epoxy or graphite/epoxy will be used as the main plate material. The material properties and ultimate allowables must be input for the material system selected.

The leading edge distance of the main plate must be defined since values of  $F_{SU}$  are not tabulated for composite laminates. Such information would be very valuable to the current effort since excessively large edge distances result in low joint efficiencies. A value of  $E/D = 4.0$  will be used for the leading edge of the composite main plate. A value of  $E/D$  may be calculated for the



for the leading edge of the splice plate from the value of  $F_{SU}^S$  and the maximum bolt load carried by the last row of bolts, which will be determined using the bolt load partitioning analysis.

In summary the required input data is:

D	Bolt diameters
$F_{SU}^B$	Effective shear strength of bolt material
F	Maximum load to be carried per column of bolts
N	Number of bolts per column
$F_{TU}^S$	Effective tension strength of splice plate material
$F_{SU}^S$	Effective shear-out strength of splice plate material
$F_{BRU}^S$	Effective bearing strength of splice plate material
$E_S$	Splice plate modulus
$E_{11}, E_{22},$ $G_{12}, \nu_{12}$ }	Composite material lamina properties
$\sigma_{1ut}, \sigma_{2ut}$ $\sigma_{1uc}, \sigma_{2uc}$ $\epsilon_{1ut}, \epsilon_{2ut}$ $\epsilon_{1uc}, \epsilon_{2uc}$ $\tau_{12u}, \gamma_{12u}$ }	Lamina ultimate allowables (Main plate)

#### 3.2.6.2.2 Design Variables

In this section seventeen parameters are defined which can be used in conjunction with the input data to completely define a given joint. If restrictions are not imposed on the design once the input data is determined, the seventeen parameters would represent seventeen design variables. If restrictions are imposed the number of design variables would be less than seventeen.

At the beginning of the program, just after the input data is read in, flags will be used to indicate which of the seventeen parameters are to be predefined. The values of these predefined parameters will then be read in as additional input data. The remaining parameters will represent the design variables.

In some cases the design variables have been restricted to a certain range of allowable values. Optimization routines require a well defined design space within which they may search for local minima. Therefore, where limits have not been specified for design variables it is up to the programmer to do so.

The program has been restricted to the  $(0/\pm\alpha/\pm\beta/90)$  class of laminates;  $\alpha$  and  $\beta$  being design variables. The values of  $\alpha$  and  $\beta$  are restricted to the range  $15^\circ$  to  $75^\circ$ . It may be desirable later to restrict the possible values of  $\alpha$  and  $\beta$  to integer values, but for the present work integer optimization procedures will not be used. It will also be required that at least one ply of each of the four lamina orientations be present in each proposed design. In this

manner we are assured that fiber failures will accompany laminate failure regardless of failure mode. A design where  $\alpha$  and  $\beta$  are set equal,  $(0/\pm\alpha/90)$ , is also acceptable since fiber failures still must accompany all possible laminate failure modes. The total thicknesses of the various lamina orientations are assumed to be, at most, linear functions of  $x$ , the position along the joint. More complicated lay up patterns will not be included in the present study.

Consider the joint design shown in Figure 14. The seventeen possible design variables are indicated on the figure. As previously mentioned the values of  $(E/D)_m$  and  $(E/D)_s$  will be specified by the program. If a joint is being designed which will consist of a number of identical columns of bolts the widths of the main plate and splice plates must be equal and constant along their lengths. The designer must input such information as described above.

In summary the seventeen possible design parameters are as follows:

<i>Parameters</i>	<i>Description</i>	<i>Range</i>
$\alpha, \beta$	Lamina orientations	$15^\circ \rightarrow 75^\circ$
$W^s(0), W^s(L)$	Width of splice plate at $x=0, L$	$3D \rightarrow$
$t^s(0), t^s(L)$	Thickness of splice plate at $x=0, L$	$\rightarrow$
$W^m(0), W^m(L)$	Width of main plate at $x=0, L$	$3D \rightarrow$
$L^m$	Joint length	$[2(N-1)+4]D \rightarrow$
$t_0(0), t_0(L)$	Thickness of $0^\circ$ laminae at $x=0, L$	1 ply
$t_{90}(0), t_{90}(L)$	Thickness of $90^\circ$ laminae at $x=0, L$	1 ply $\rightarrow$
$t_{\pm\alpha}(0), t_{\pm\alpha}(L)$	Thickness of $\pm\alpha^\circ$ laminae at $x=0, L$	1 ply $\rightarrow$
$t_{\pm\beta}(0), t_{\pm\beta}(L)$	Thickness of $\pm\beta^\circ$ laminae at $x=0, L$	1 ply $\rightarrow$

the lower limits on the widths and length measurements are based on a minimum separation of free surfaces of one diameter. These values may be changed by the programmer if desired. The upper and lower limits which have not been specified must be provided by the programmer.

#### 3.2.6.2.3 Design Constraints

Once a design is proposed it must be loaded to design ultimate. The joint failure criterion, Section 3.2.5, must then be applied to determine whether or not the proposed design can indeed carry the design ultimate load, as required. In order to automate the process of examining the joint for possible failure at the  $i^{\text{th}}$  critical location an equality constraint,  $F(i)$ , must be defined. For the in-house pattern search optimization routine, the equation must be written in such a form that  $F(i) \leq 0$  if failure is not indicated. If failure is predicted to occur, then  $F(i) > 0$ . As previously mentioned a penalty function is added to the weight function when  $F(i) > 0$ . To minimize the total weight of the joint the design must move in design space in a direction which tends to reduce the penalty functions.

Consider the possible failures which could occur at each hole along the joint. They are:

- (1) Bolt failure in shear.
- (2) Splice plate failures in bearing, shear-out, or net tension.
- (3) Main plate failures in bearing, shear-out, net tension, splitting, bending tear-out, or combination modes.

An inequality constraint equation must be written for each of the possible failure initiation sites.

To test for bolt failure in shear we calculate the maximum shear stress,  $\tau$ , acting on the bolt cross sectional area.

$$\tau = P_B / (\pi R^2) \quad (13)$$

$P_B$  represents the bolt load acting at the hole of interest. The magnitude of  $P_B$  is determined via the bolt load partitioning analysis, Section 3.2.4.

To insure against a bolt failure in shear we require that  $\tau \leq F_{SU}^B$ . Stating this in the form of a valid inequality constraint we have:

$$F(1) = P_B / (\pi R^2) - F_{SU}^B \quad (14)$$

Similarly, to insure against splice plate failures at a given hole in bearing, net tension or shear-out we have respectively:

$$F(2) = P_B / Dt - F_{BRU}^S \quad (15)$$

$$F(3) = (P_B + P_S) / t(S - D) - F_{TU}^S \quad (16)$$

$$F(4) = P_B / 2tE - F_{SU}^S \quad (17)$$

The skin load,  $P_S$ , is calculated from equation (12). The values of  $t$ ,  $E$ , and  $S$  for a given bolt bearing model are determined as was shown in Figure 4.

Now consider the possible composite main plate failures. Once a stress analysis is performed, checks for possible failure initiation must be made at four locations (possibly only three if

$\alpha$  equals  $\beta$ ) around each hole as discussed in Section 3.2.5. Bearing failures may or may not be considered desirable. If they are, the load redistribution procedures discussed in Section 3.2.5 can be built into the computer logic. To simplify the following discussion assume that bearing failures are undesirable.

Thus, if at  $\theta = 0^\circ$ , a matrix failure occurs in the  $0^\circ$  laminae ( $\sigma_2/\sigma_{2ut} \geq 1.0$ ) a bearing failure would be predicted to occur when  $\sigma_1/\sigma_{1uc} = 0.65$ . If matrix failures do not occur during loading then  $\sigma_1/\sigma_{1uc} \geq 1.0$  would be necessary for a bearing failure to occur. Since the likelihood of a bearing failure is only dependent on the stresses at  $\theta = 0^\circ$  in the  $0^\circ$  laminae, an inequality constraint equation may be written at that location of the form

$$F(5) = \begin{cases} \sigma_1/\sigma_{1uc} - 1.0 & \text{if } \sigma_2/\sigma_{2ut} < 1.0 \\ \sigma_1/\sigma_{1uc} - 0.65 & \text{if } \sigma_2/\sigma_{2ut} \geq 1.0 \end{cases} \quad (18)$$

It was postulated in Section 3.2.5 that all failure modes, except bearing, have one thing in common. They all seem to occur at locations where fibers are tangent to a hole surface. In a  $(0 / \pm\alpha / \pm\beta / 90)$  laminate fibers are tangent to the hole surface at  $\theta = 90^\circ$ ,  $(90 - \alpha)^\circ$ ,  $(90 - \beta)^\circ$ , and  $0^\circ$  in the  $0^\circ$ ,  $+\alpha^\circ$ ,  $+\beta^\circ$ , and  $90^\circ$  laminae respectively. If, at ultimate load, matrix failures have occurred at any of these hole locations load redistribution among the

various laminae must be considered. The stresses in the remaining laminae would be recalculated. A check would then be made in the laminae which are tangent to the hole surface to see whether or not the fibers have failed in tension. The inequality constraint used for this purpose is:

$$F(6) = \sigma_1 / \sigma_{1ut} - 1.0 \quad (19)$$

Equation (19) must be applied four times per hole; to the  $0^\circ$  fibers at  $\theta = 90^\circ$ , the  $90^\circ$  fibers at  $\theta = 0^\circ$ , the  $+\alpha^\circ$  fibers at  $\theta = (90 - \alpha)^\circ$ , and the  $+\beta^\circ$  fibers at  $\theta = (90 - \beta)^\circ$ . Thus, a total of nine inequality constraints must be satisfied at each and every hole.

In the past, designers have designed for net tension failures at the innermost row of bolts. An equality constraint of the form

$$\sigma_1 / \sigma_{1ut} - 1.0 = 0 \quad (20)$$

could be imposed on the stress field in the  $0^\circ$  laminae at the innermost row of holes to force the design to fail there in net tension. Such a restriction is not justified, however. When an optimum design is arrived at using the nine inequality constraint equations per hole, one of the nine equations will, in the process, be automatically forced to zero. This will specify joint failure mode and location. Net tension failures at the innermost row of bolts may not result when minimum weight designs are required.

#### 3.2.6.2.4 Design Procedures

In order to begin the design process an initial design

must be selected. The initial design must, of course, be in the design space which is defined by the upper and lower limits placed on the design variables.

If the designer has a design in mind he may use it to activate the program. Otherwise, the program will specify an initial design. The in-house pattern search routine uses a random number generator for the purpose of specifying initial values for the design variables. It may be desirable to use several random starting points, if run times are not excessively long, to check for possible local minimum in the design space.

Once an initial design is proposed the bolt load partitioning results would be used to calculate the bolt load distribution for the geometry and lamination selected. To perform such calculations the main plate and splice plate cross sectional areas as well as the main plate modulus must be defined as functions of  $x$ . Referring back to Figure 14 it can be shown that

$$A_m(x) = [(t_m(L) - t_m(0)) (x/L) + t_m(0)] [(w_m(L) - w_m(0)) (x/L) + w_m(0)] \quad (21)$$

$$A_s(x) = [(t_s(L) - t_s(0)) (x/L) + t_s(0)] [(w_s(L) - w_s(0)) (x/L) + w_s(0)] \quad (22)$$

It has been found by this investigator that a quadratic polynomial in  $x$  can be used to represent the modulus of the main plate to within a few percent when linear variations in lamina thicknesses are employed.



Thus, the modulus of the main plate may be determined at several locations along the joint using lamination theory and a second order curve of the form

$$E_m(x) = Ax^2 + Bx + C \quad (23)$$

may be fit to the resulting modulus values. The values of A, B, and C will be determined automatically by an internal curve fitting subroutine for the proposed design.

The only remaining unknowns which are needed to calculate the bolt load distribution are the coordinate locations of the N bolts. Since L, the joint length, and  $(E/D)_m$ , the leading edge distance of the composite main plate, are known, we may express the N bolt locations as:

$$X(I) = \frac{I-1}{N-1} \left[ L - \left( \frac{E}{D} \right)_m \right] \quad \text{where } I = 1, N \quad (24)$$

The bolt loads could then be calculated using equations (4) and (5) or equations (9) and (11).

The next step in the design procedure is to perform a row by row stress analysis of the proposed design. The column of bolts is broken down into individual bolt bearing specimens as shown in Figure 4. The value of  $P_{sj}$  would be calculated using equation (12). Thus, each bolt bearing model is acted on by a bolt load,  $P_{Bi}$ , and a skin stress,  $\sigma_{sj} = P_{sj}/St$ . Since a finite element solution of each bolt bearing specimen is too costly the following procedures will be followed.

The problem of an infinite plate containing a circular cutout which is loaded as shown in Figure 13a will be solved. This investigator's solution [11] to the problem illustrated in Figure 13b will be added to the solution of the problem illustrated in Figure 13c, which is presented in [12]. Corrections to the stress concentration factors induced at  $\theta = 90^\circ$  and  $\theta = 0^\circ$  will be made to account for the effects of finite specimen size using the results presented in [13] and [14]. Corrections to the average laminate stresses along the circular boundary from  $\theta = 0^\circ$  to  $\theta = 90^\circ$  can then be estimated.

The corrected average laminate stresses would be transformed to lamina stresses and checks would then be made to see if the design constraints discussed in Section 3.2.6.2.3 were satisfied. Penalty functions would be assigned to the weight function for each of the constraint equations which was not satisfied and the total joint weight would then be calculated. The optimization procedure would determine a preferred path and select a new design along that path which would have a lower total joint weight while more closely satisfying all the imposed design constraints.

The entire process, beginning with the calculation of bolt loads for the new design would be repeated until the design constraints were all satisfied and a local minimum weight were achieved.

Since the design procedure uses isotropic correction factors to account for finite specimen size there is, of course, some

doubt concerning the actual failure behavior of the proposed optimum design. Therefore, it may be desirable to perform a complete stress analysis for the proposed optimum design to see how closely the predicted failure behavior would agree with the desired failure behavior. The analysis method described in Section 3.2.5 would be used if the final check were to be made.

### 3.2.7 Areas of Future Work

Several important questions have been raised regarding the solution technique thus far which deserve mention and in most cases warrant further investigation. The first involves the basically different failure mechanisms which can occur in a given joint made of composite materials. Net tension failures appear to behave as brittle failures once a very small but finite region of localized laminate destruction occurs. Bearing failures and shear-out failures, on the other hand, do not occur unless extensive laminate damage has resulted during loading. Thus bearing and shear-out failures behave in a relatively ductile manner. Additional analytical and experimental work must be done to understand the various failure mechanisms which occur in composite joints before truly optimum designs can be achieved. The results of the proposed geometrically similar bolt bearing specimen testing program should bring us closer to such an understanding.

In metals, effective bearing strengths,  $F_{BRU}$ , and effective shear-out strengths,  $F_{SU}$ , have been experimentally determined and are used in the design process to specify such parameters as leading edge distances. Similar information is generally not available for composites due to the number of possible laminates which could be used for design purposes. Such "material properties" would be invaluable, however, in the design of composite joints and deserve further investigation.

Similarly, a lack of data concerning the effects of finite size on stress concentrations induced at circular cut-outs in composite plates has forced us to predict these effects from available isotropic data. Making corrections from isotropic data results in a conservative design and is, therefore, partially satisfactory. The need for correction factors could be eliminated, however, if the in-house two dimensional anisotropic integral equation program developed by Dr. T. A. Cruse could be built into the optimization program in such a way as to not result in excessive computer run times. One other technique would be to derive the necessary correction factors for various laminates using the integral equation program and use such data in place of the isotropic correction factors which are now being used. Both possibilities are presently being investigated.

The following questions will also be considered:

- (1) Are uniform bolt load distributions and net tension failures at the innermost row of bolts requirements for optimum joint designs?
- (2) Is it advantageous to use the displacement matching technique rather than the strain matching technique to predict bolt load distributions?
- (3) How should load redistribution in a joint be handled once a bearing failure occurs either in the main plate or splice plates?
- (4) Is the uncertainty regarding the cross term in the distortional energy failure criterion, as discussed in [15], a major problem to be considered? A preliminary investigation performed during

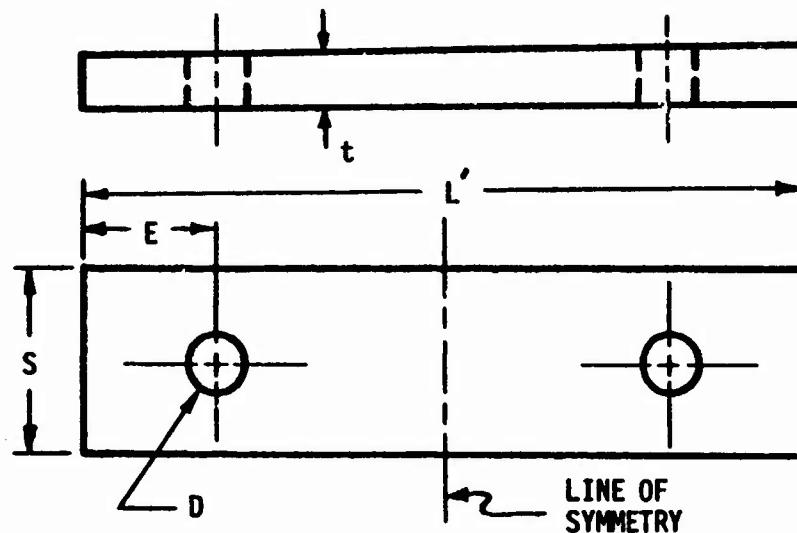
the contract period has shown that in certain cases the cross term may be the most important term in the energy relationship. These questions will be pursued as part of this investigator's doctoral thesis during the current academic year.

### 3.2.8 References

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B/E  
(0/±45/90)

Table 1. Initial Bolt Bearing Specimen Design

Spec. No.	D	E	S	$L'$	t	No. Plies	Predicted Failure Loads(lb)	
							Equations(4)	Finite Elements
1	0.125	0.31	0.53	1.875	0.08	16	647	647
2	0.250	0.62	1.06	3.750	0.16	32	2,587	2,587
3	0.375	0.93	1.59	5.625	0.24	48	5,820	5,820
4	0.500	1.24	2.12	7.500	0.32	64	10,350	10,350

Table 2. Revised Bolt Bearing Specimen Design

Spec. No.	D	E	S	$L'$	t	No. Plies	Predicted Failure Loads(lb)	
							Equations(4)	Finite Elements
1	0.125	0.50	0.375	2.25	0.08	16	400	625
2	0.250	1.00	0.750	4.50	0.16	32	1,600	2,500
3	0.375	1.50	1.125	6.75	0.24	48	3,600	5,620
4	0.500	2.00	1.500	9.00	0.32	64	6,400	10,000

Table 3. Description of the Six Specimens Selected for Analysis

Spec. No.	Lamination	Splice Plate Material	Loading	Rows of Bolts	Bolts per Row	Failure Mode	Ultimate Load(lb)
1	B/E, $0_4/\pm 45$	D6-AC Steel	SS	6	4	SP	94,200
2	B/E, $0_4/\pm 45$	D6-AC Steel	SS	5	4	SP	115,500
3	B/E, $0_4/\pm 45$	6-4 T1	SS	5	4	SP	110,400
4	B/E, $0_4/\pm 45$	D6-AC Steel	SS	4	4	SP	125,400
5	B/E, $0_2/\pm 45$	D6-AC Steel	DS	4	4	T	189,000
6	G/E, $0/\pm 45^1$	6-4 T1	SS	6	2	T	74,800

Nomenclature:

SS Single Shear  
DS Double Shear  
B/E Boron-Epoxy  
G/E Graphite-Epoxy  
SP Splitting  
T Tension

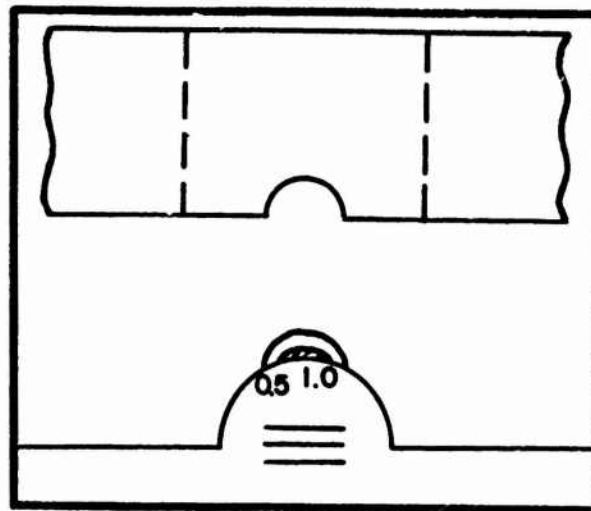
Notes:

(1) Lamina thicknesses vary linearly along the specimen length from  $0_2/\pm 45$  at the first row of bolts to  $0_2/\pm 45_3$  at the last row of bolts.

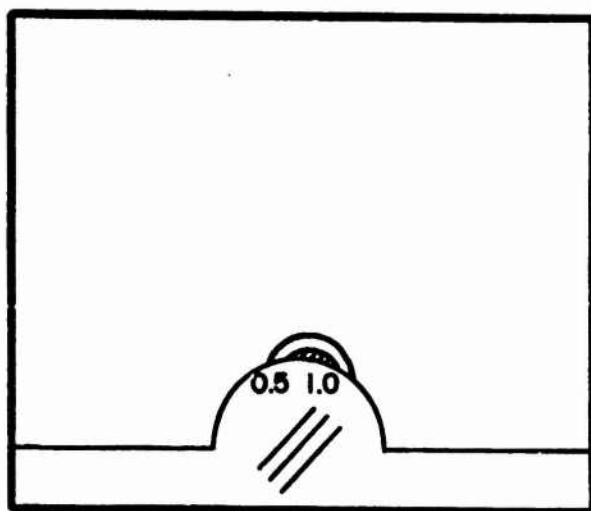
Table 4. Summary of Significant Data from Figures 7 thru 12

Spec. No.	Row No.	Bearing, $\theta=0^\circ$			Tension, $\theta=90^\circ$		Splitting, $\theta=\pm 45^\circ$			
		0° Lamina			0° Lamina		+45° Lamina		-45° Lamina	
		$\frac{\sigma_1}{\sigma_{1uc}}$	$\frac{\sigma_2}{\sigma_{2ut}}$	DIST	$\frac{\sigma_1}{\sigma_{1ut}}$	DIST	$\frac{\sigma_1}{\sigma_{1ut}}$	DIST	$\frac{\sigma_2}{\sigma_{2ut}}$	DIST
1	First	0.07	C*	0.02	1.05	1.50	0.21	0.10	0.83	1.10
	Last	0.49	2.50	5.80	1.33	2.00	1.50	2.50	1.65	3.20
2	First	0.10	0.04	0.01	0.91	1.00	0.42	0.25	0.80	0.75
	Last	0.50	2.30	5.10	1.11	1.41	1.30	2.10	1.55	2.60
3	First	0.07	0.05	0.01	0.83	0.90	0.63	0.47	0.79	0.71
	Last	0.47	2.50	6.00	1.34	1.45	1.43	2.50	1.73	3.00
4	First	0.13	0.29	0.11	0.85	0.85	0.58	0.40	0.69	0.55
	Last	0.52	2.80	7.95	1.07	1.51	1.55	2.70	1.90	3.75
5	First	0.30	0.56	0.42	1.29	2.00	0.79	0.90	0.92	0.90
	Last	0.60	2.00	4.00	1.02	1.41	1.15	1.65	1.21	1.88
6	First	0.17	0.04	0.03	0.90	0.95	0.57	0.44	0.75	1.12
	Fifth	0.37	0.26	0.21	1.03	1.20	0.63	0.60	0.83	1.15
	Sixth	0.80	1.00	1.90	0.80	1.00	0.63	0.65	0.95	0.95

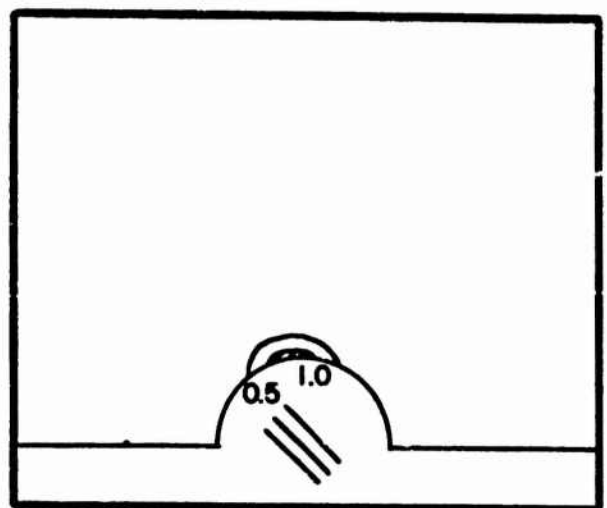
\*C Compressive stress



0° LAMINA

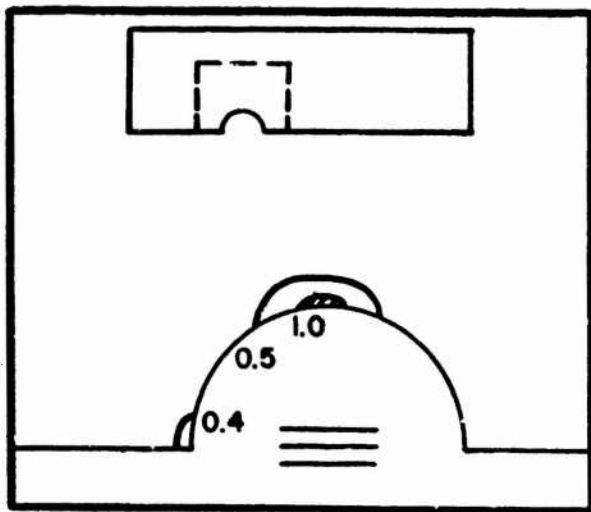


+ 45° LAMINA

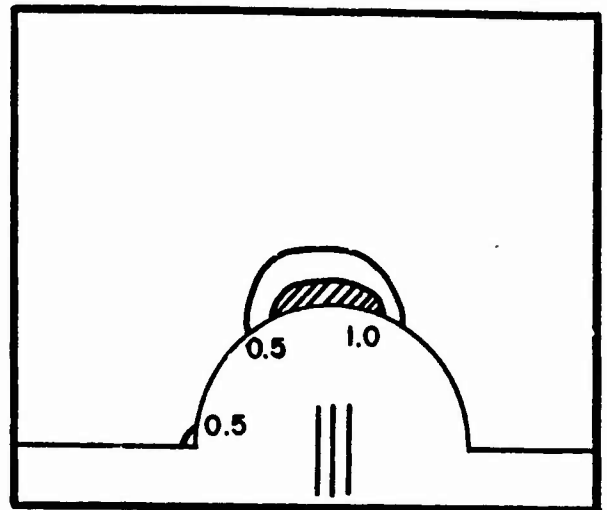


- 45° LAMINA

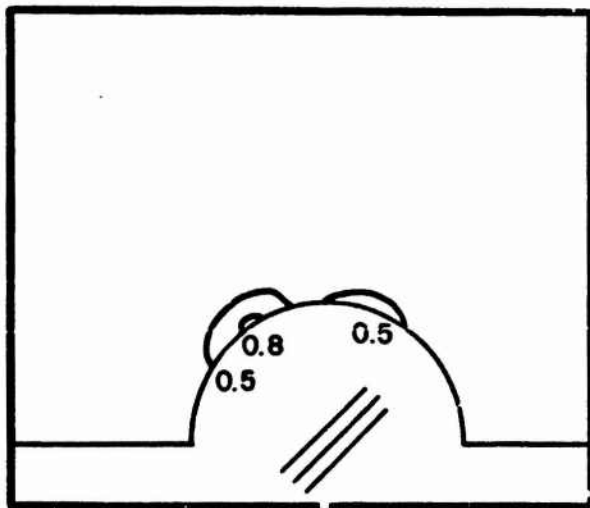
Figure 1. Distortional Energy Contour Plots for the 1.0" Diameter, Anisotropic Tension Coupon at the Experimental Failure Load.



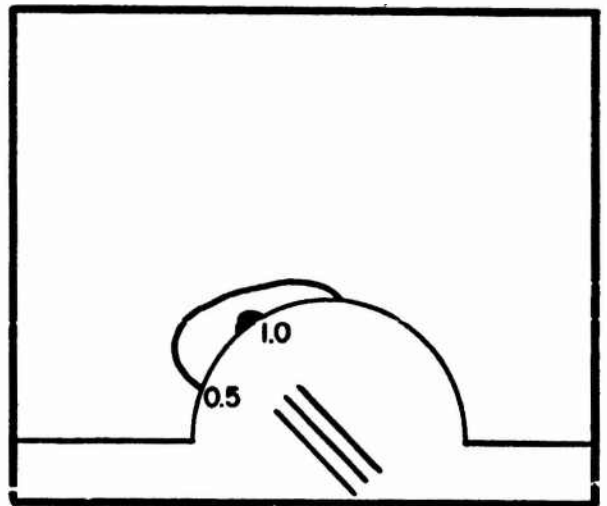
0° LAMINA



90° LAMINA

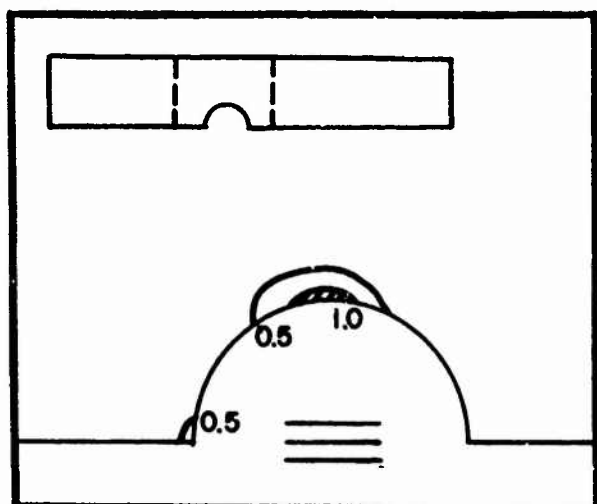


+45° LAMINA

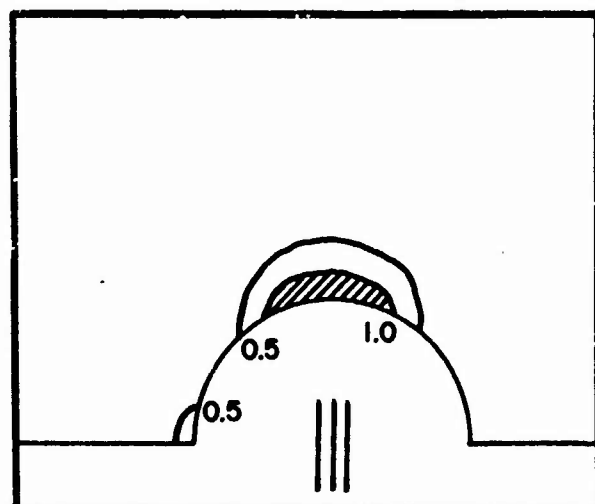


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Figure 2a. Distortional Energy Contour Plots for the Initial Bolt Bearing Specimen Design at the Predicted Failure Load.



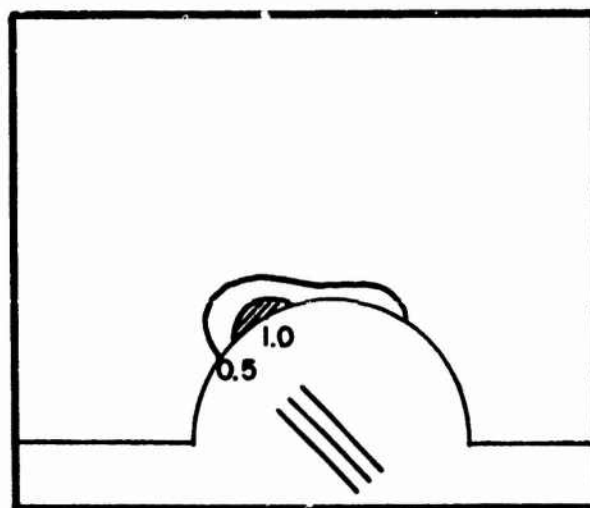
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+45° LAMINA



-45° LAMINA

Figure 2b. Distortional Energy Contour Plots for the Revised Bolt Bearing Specimen Design at the Predicted Failure Load.

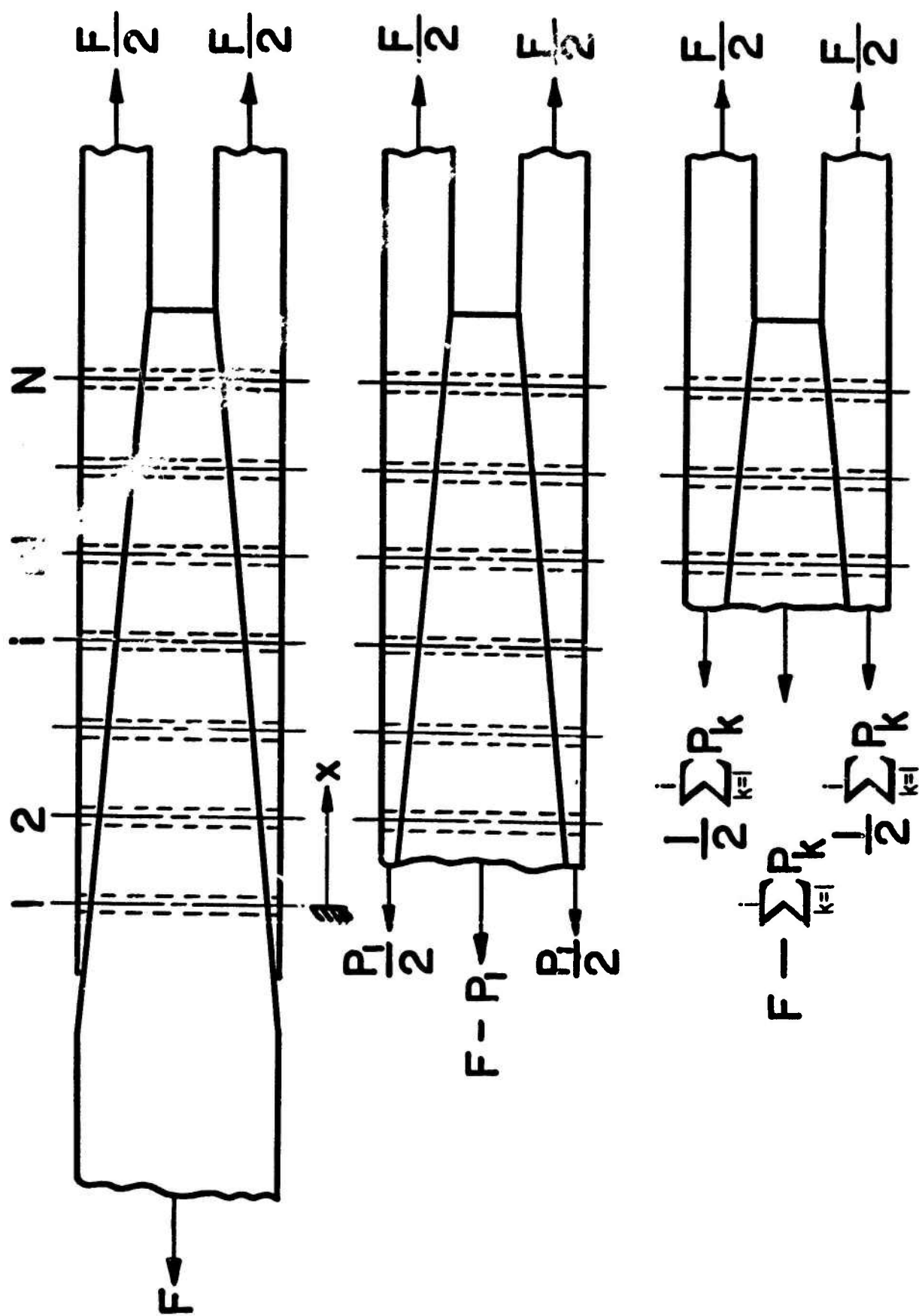


Figure 3. Double Shear Joint Configuration.

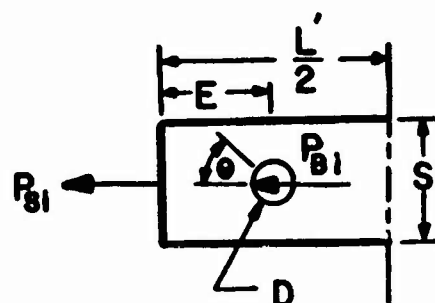
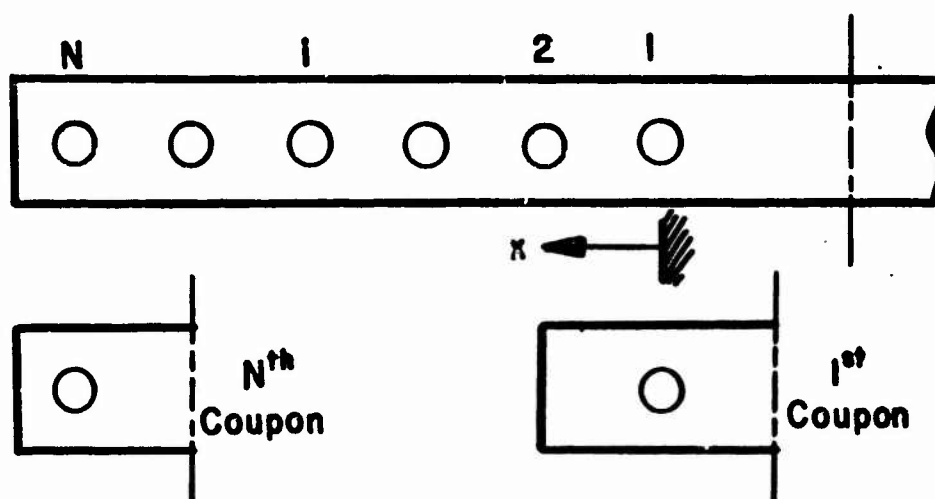
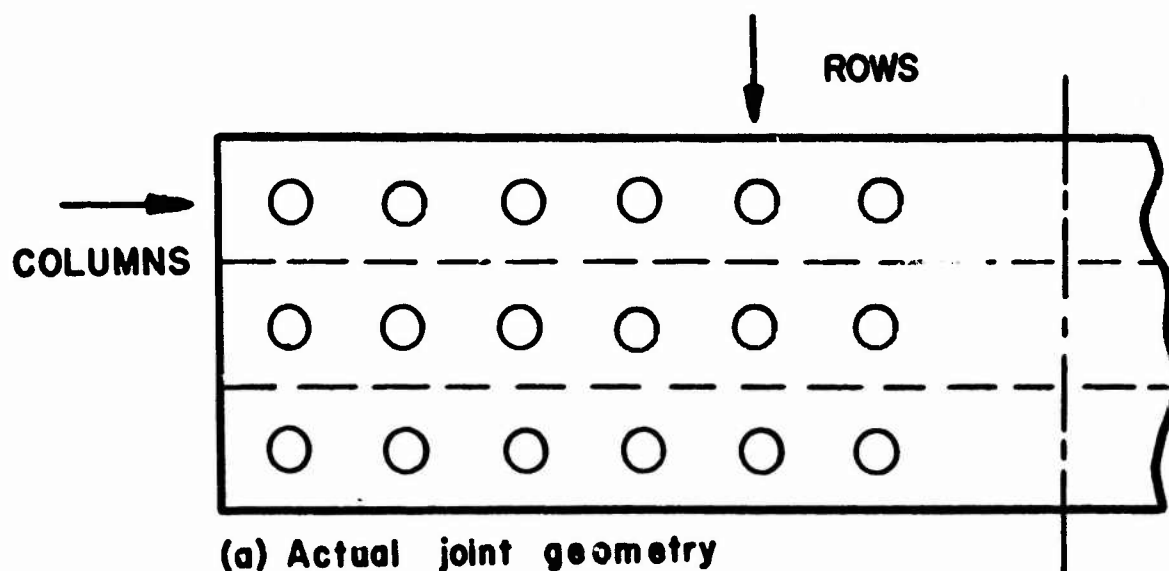
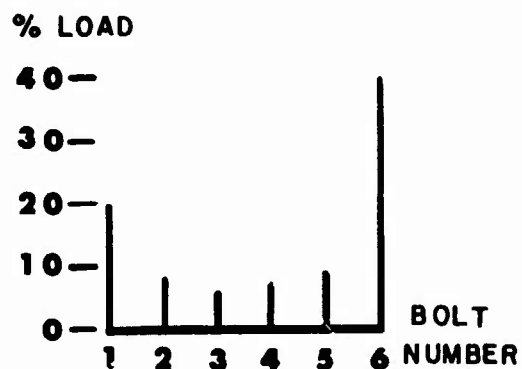
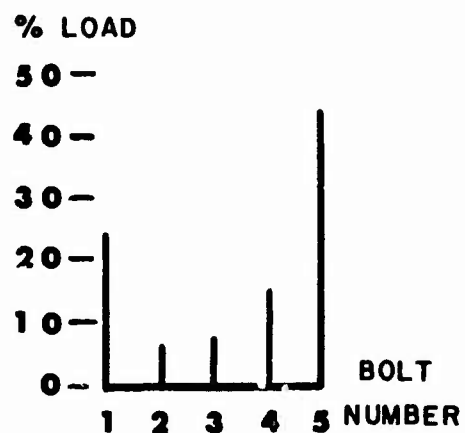


Figure 4. Bolt Bearing Modeling Procedure for Joints.

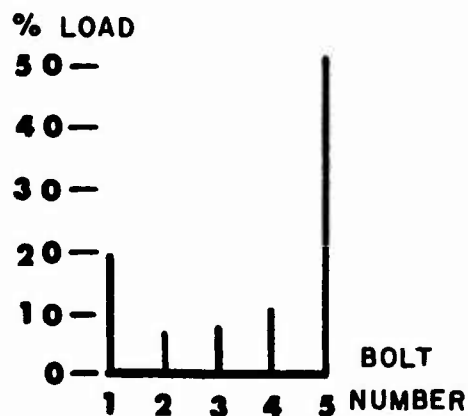




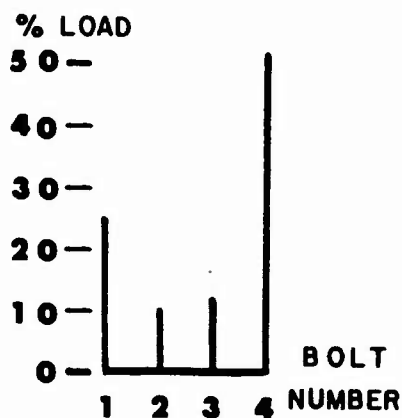
(a) Specimen 1



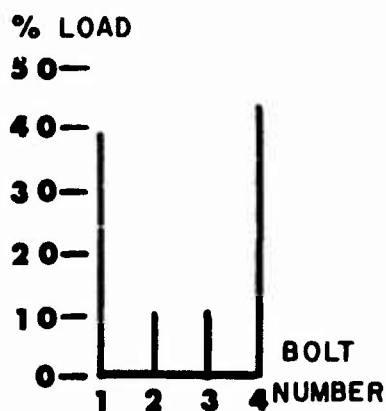
(b) Specimen 2



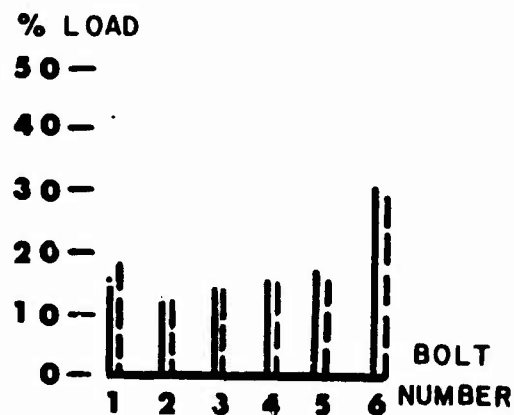
(c) Specimen 3



(d) Specimen 4

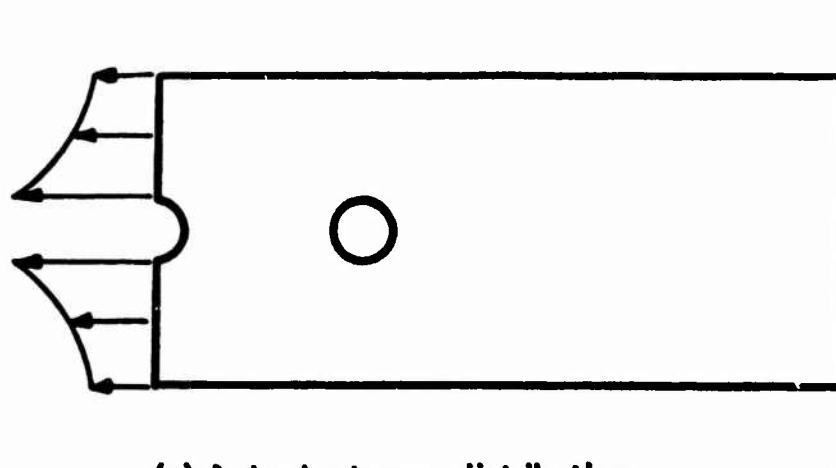


(e) Specimen 5

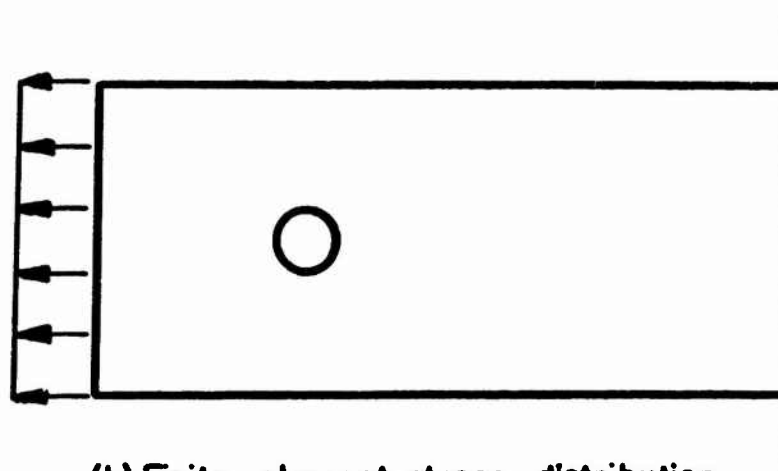


(f) Specimen 6

Figure 5. Bolt Load Distributions for Specimens 1 thru 6.

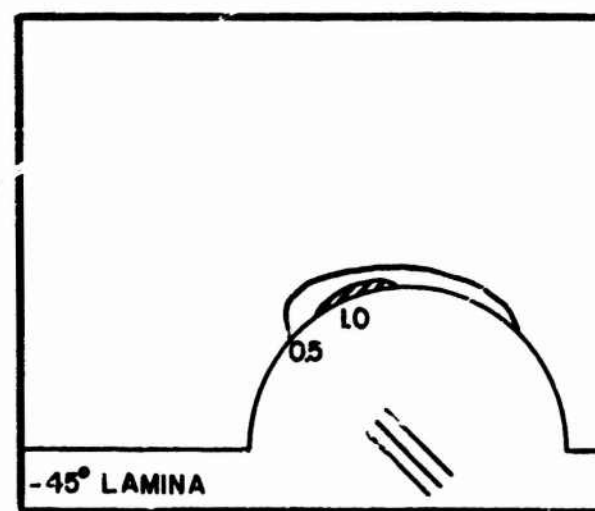
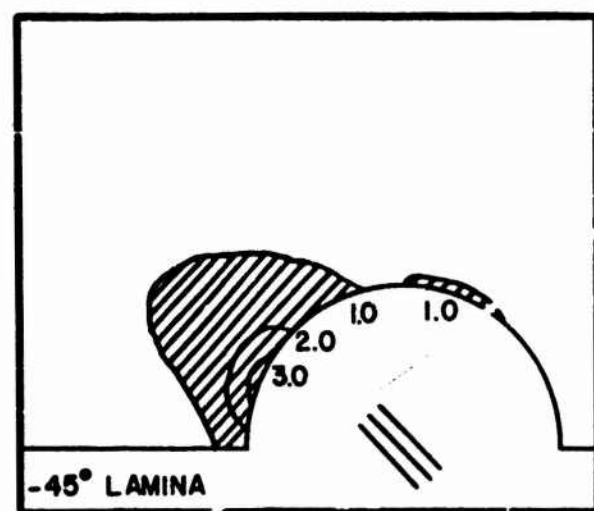
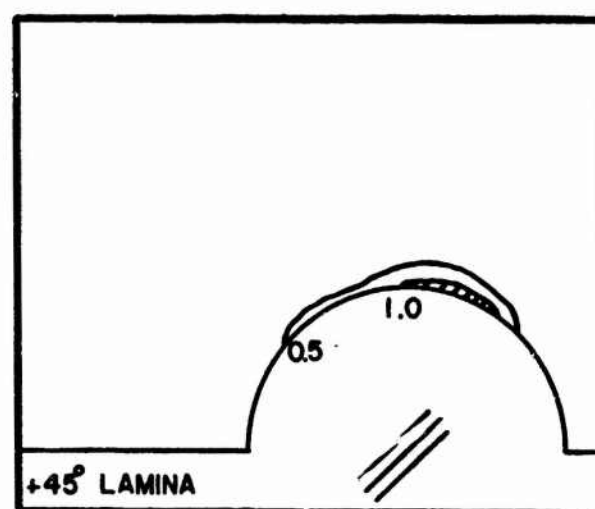
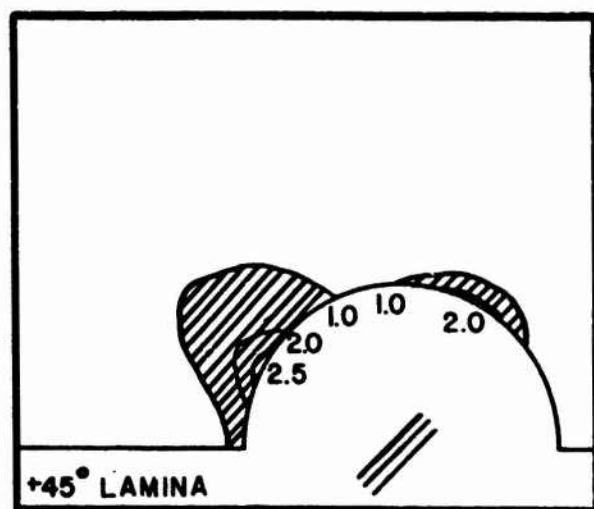
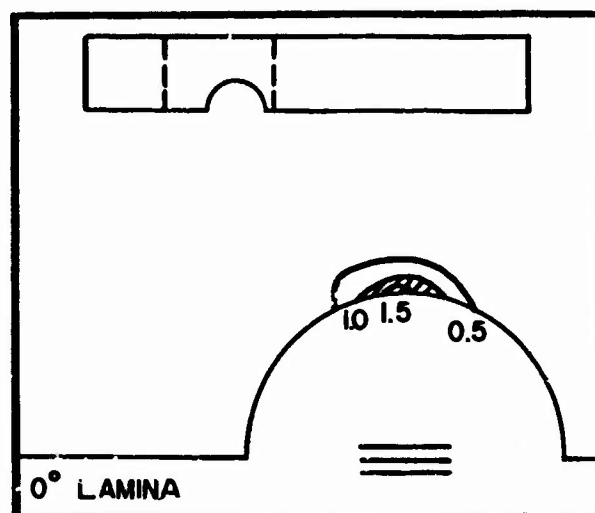
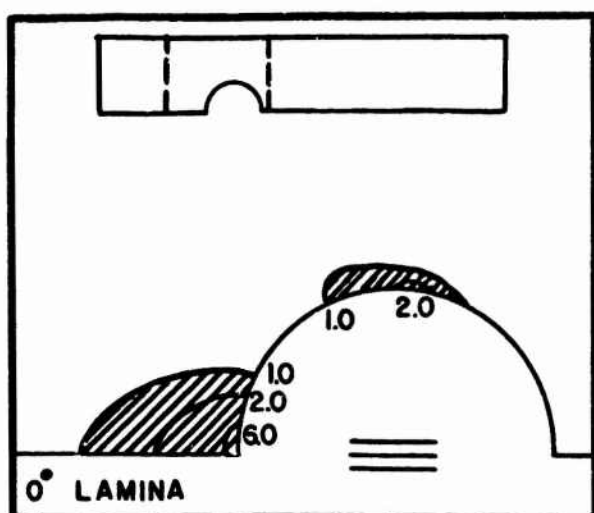


**(a) Actual stress distribution**



**(b) Finite element stress distribution**

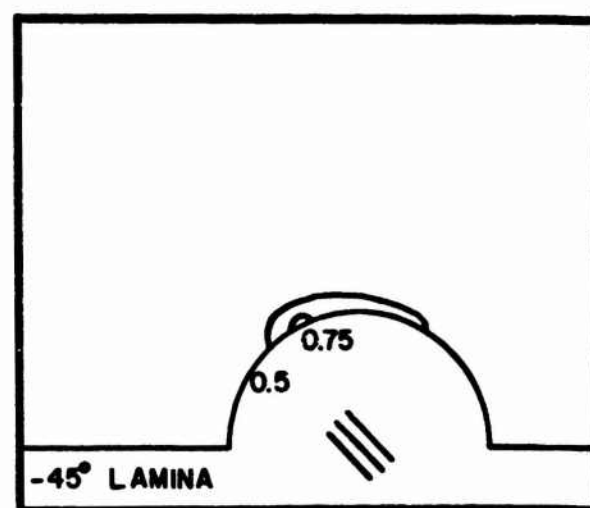
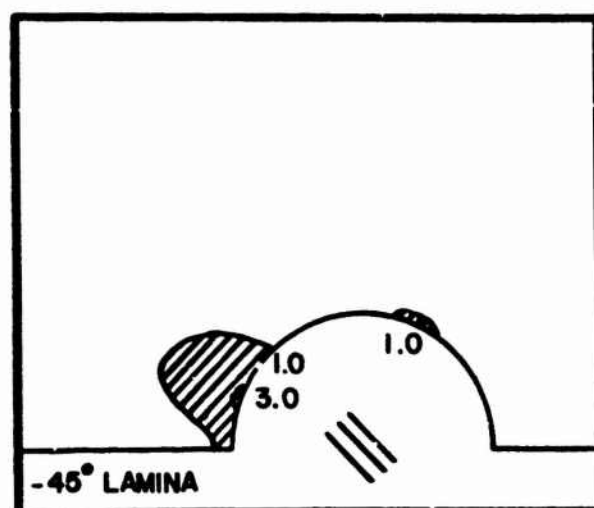
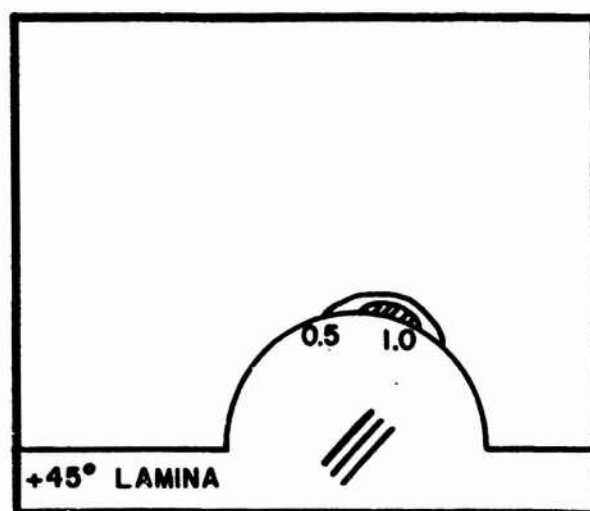
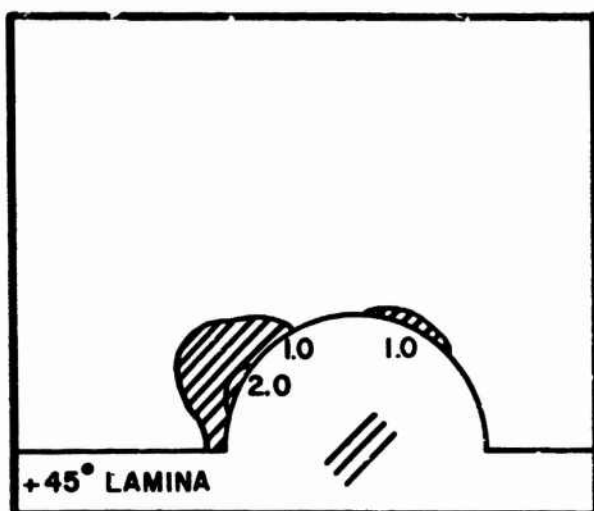
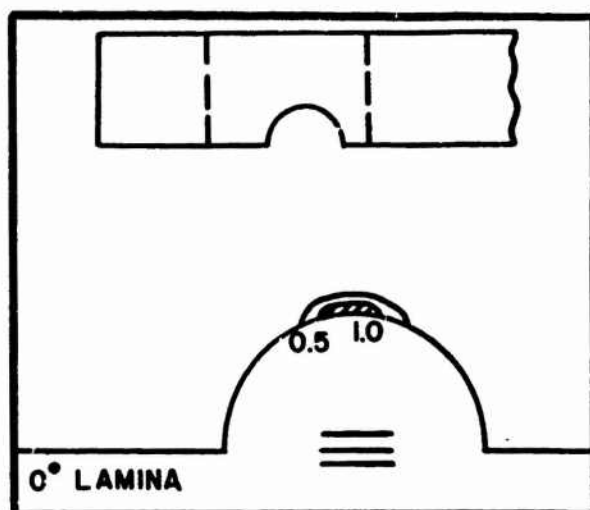
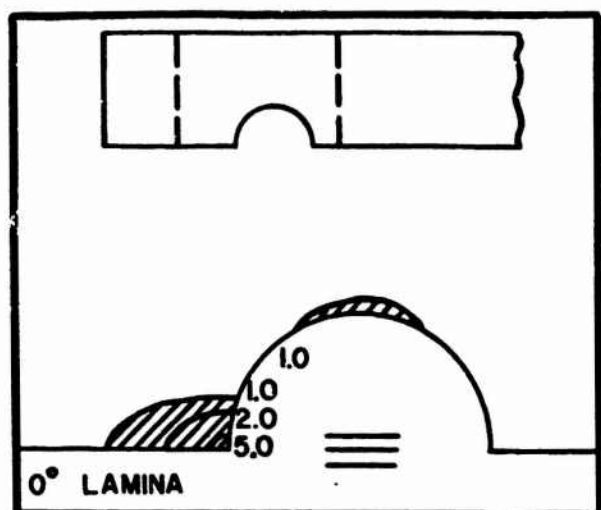
**Figure 6. Skin Stress Boundary Conditions at the Leading Edge of a Bolt Bearing Model.**



LAST ROW

FIRST ROW

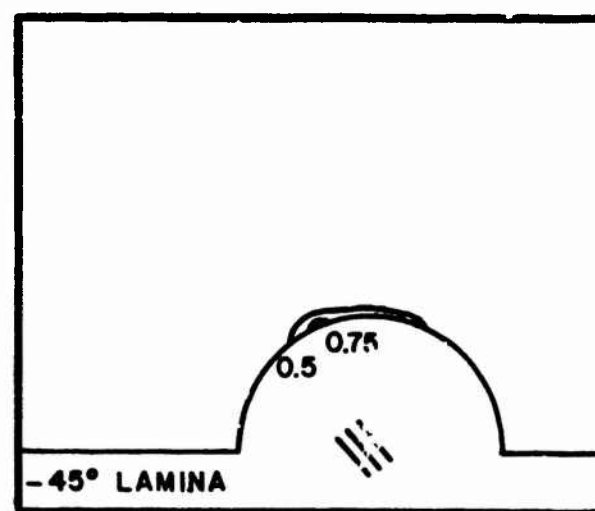
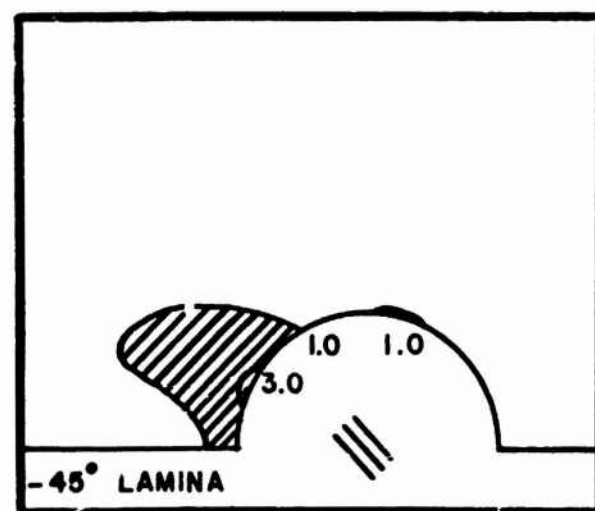
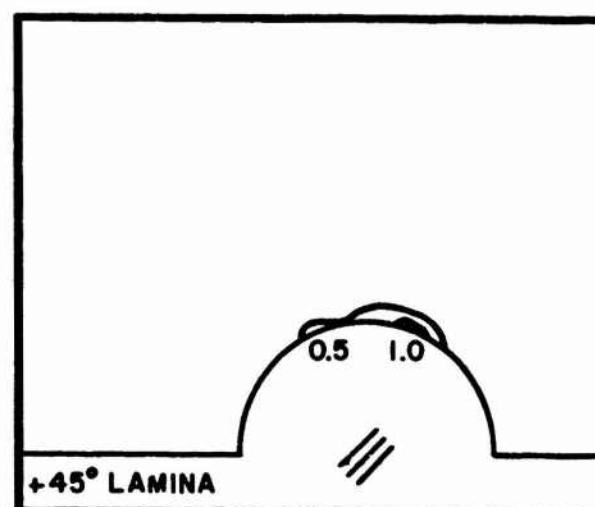
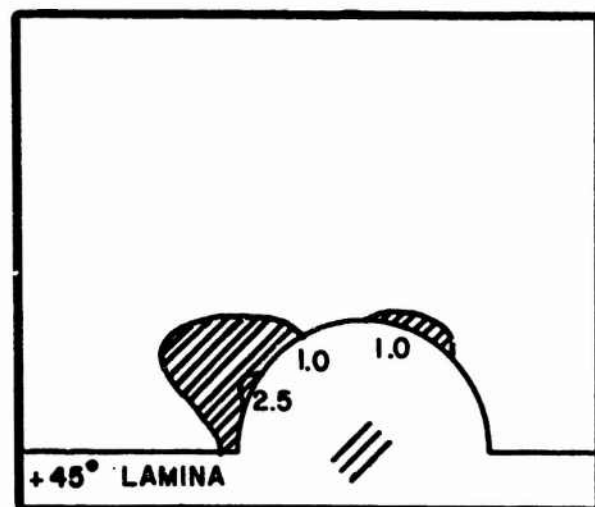
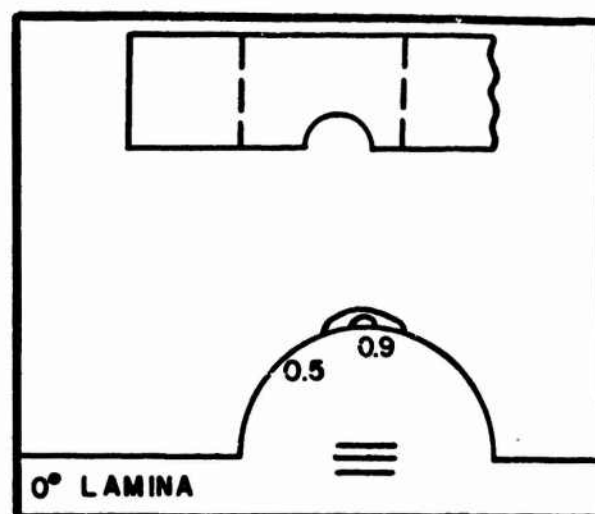
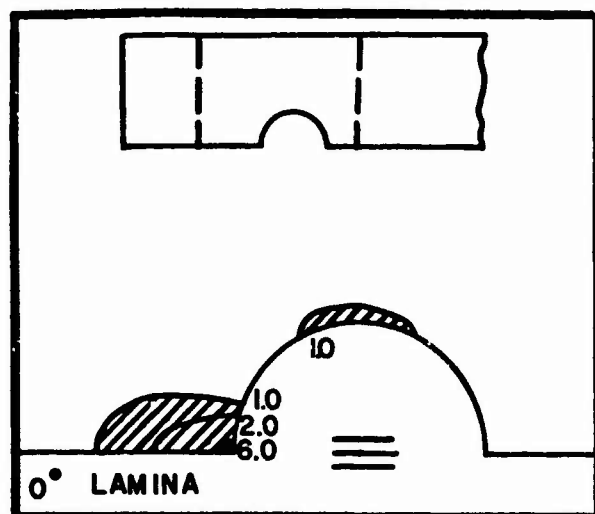
Figure 7. Specimen 1: Distortional Energy Contour Plots for the Experimental Failure Load



LAST ROW

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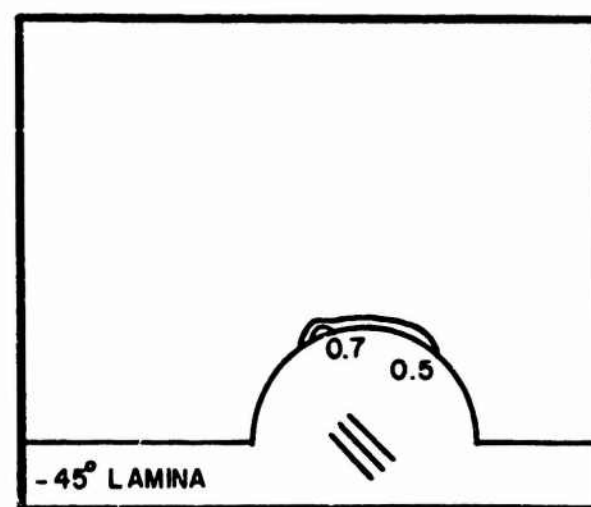
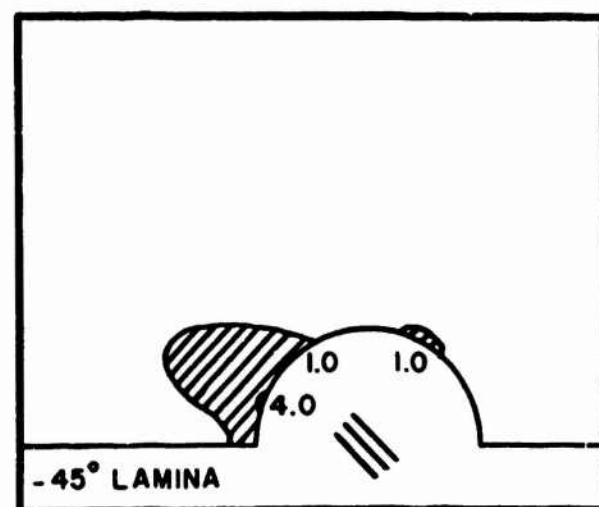
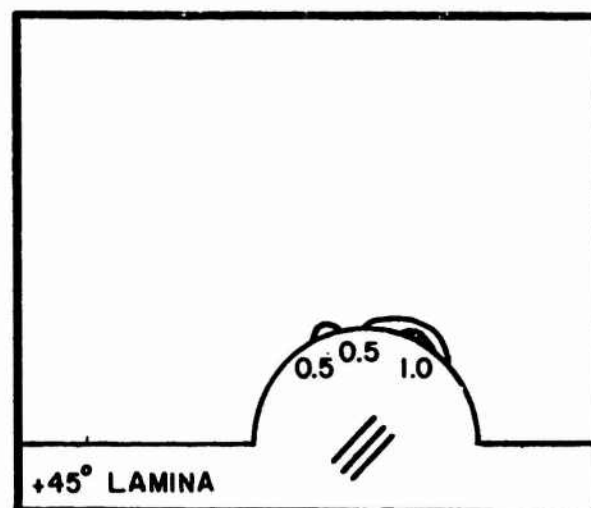
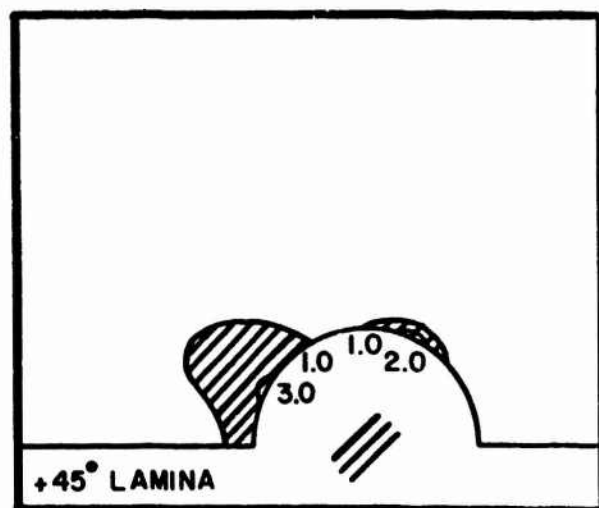
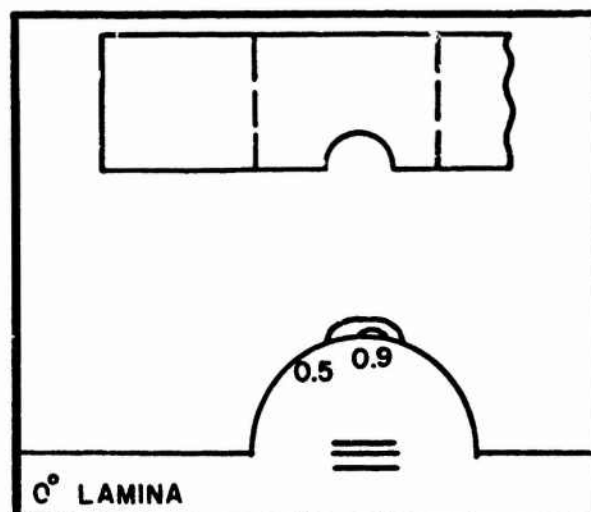
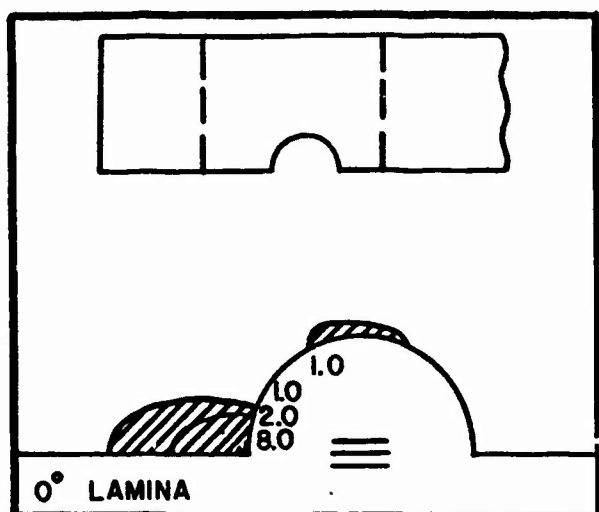
Figure 8. Specimen 2: Distortional Energy Contour Plots for the Experimental Failure Load. 133



LAST ROW

FIRST ROW

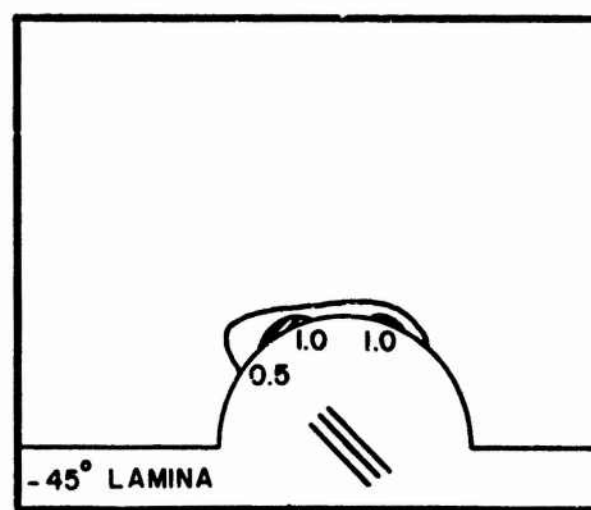
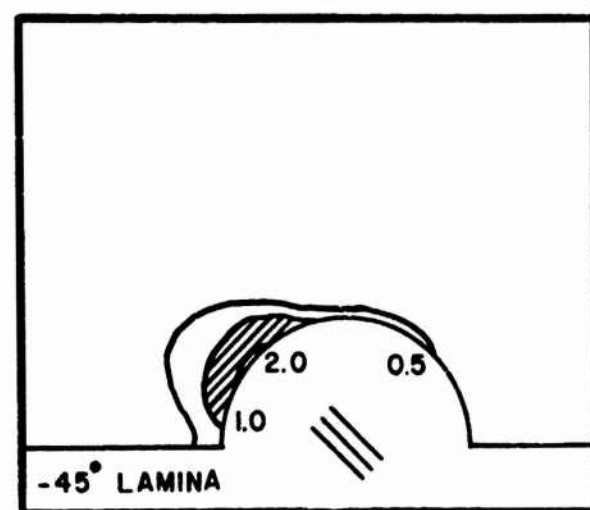
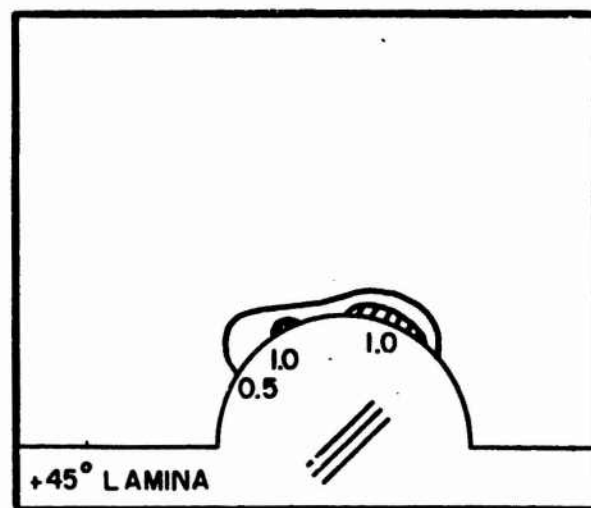
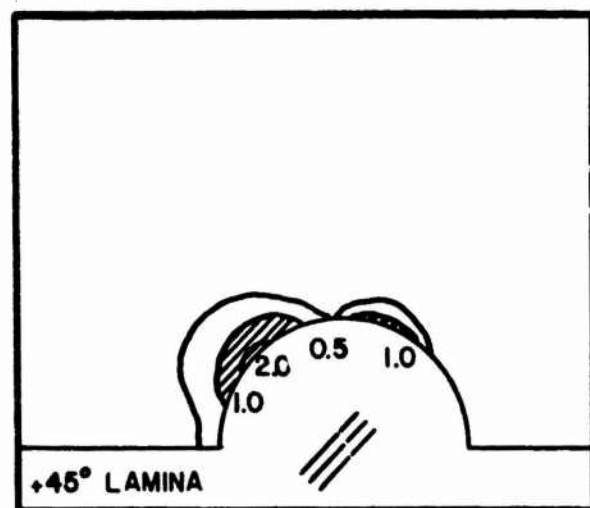
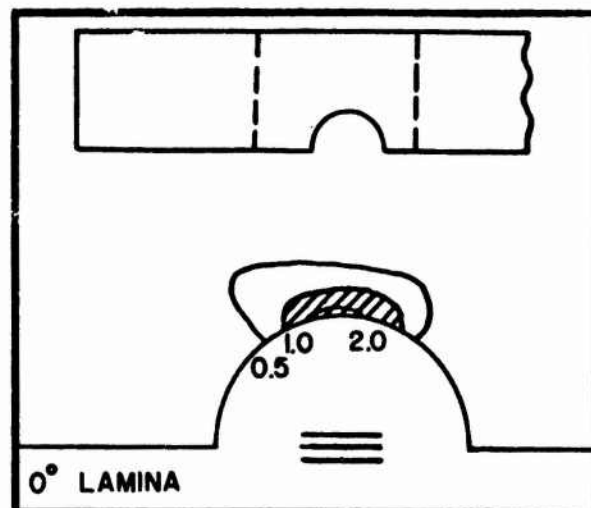
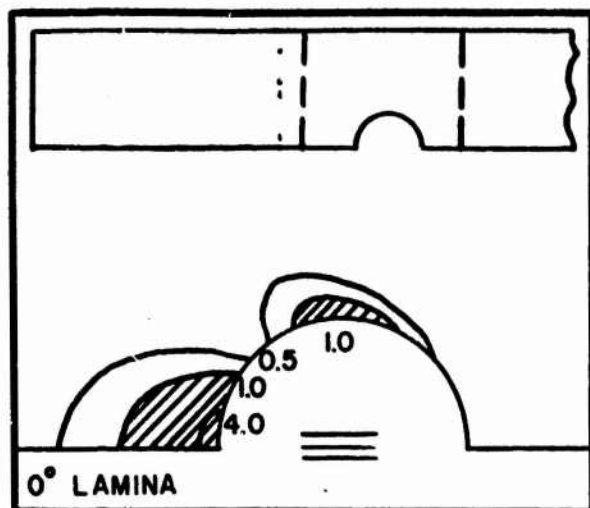
Figure 9. Specimen 3: Distortional Energy Contour Plots for the Experimental Failure Load.



LAST ROW

FIRST ROW

Figure 10. Specimen 4: Distortional Energy Contour Plots for the Experimental Failure Load.



LAST ROW

FIRST ROW

Figure 11. Specimen 5: Distortional Energy Contour Plots for the Experimental Failure Load.

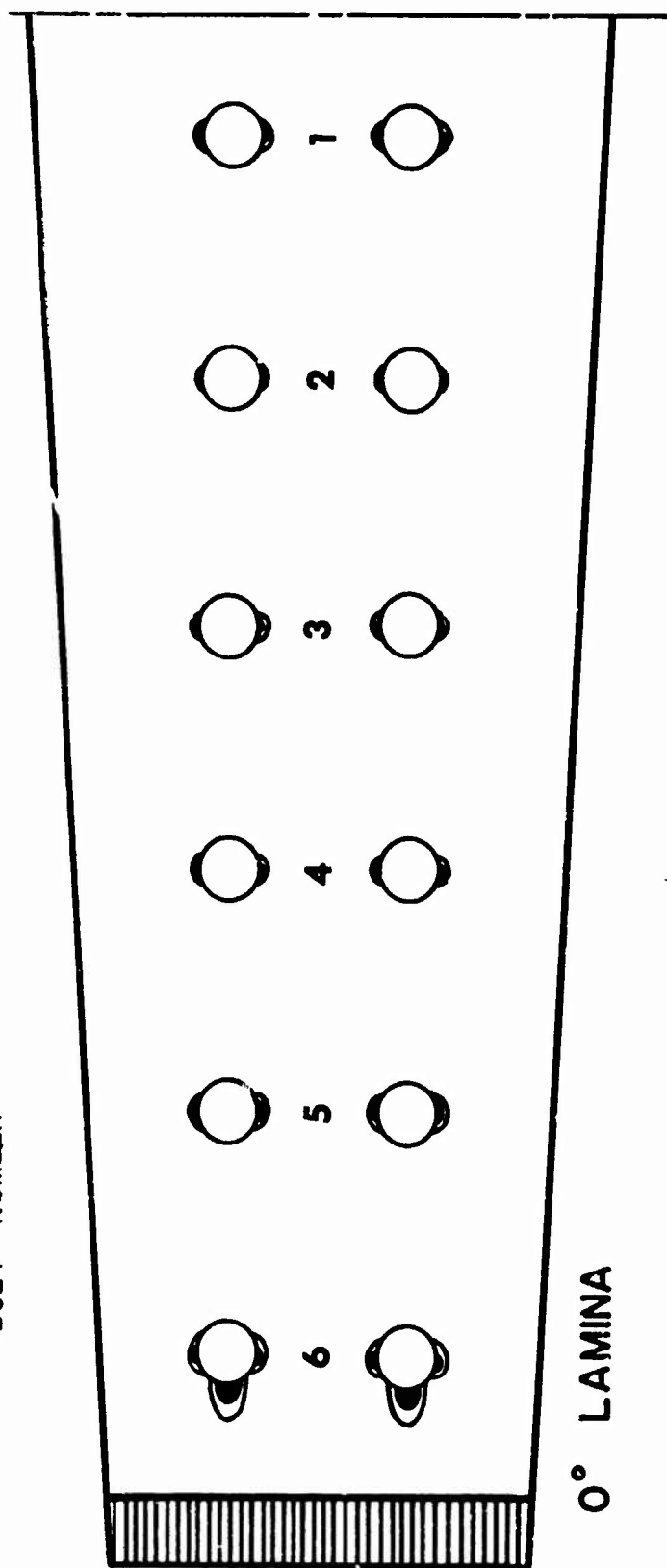
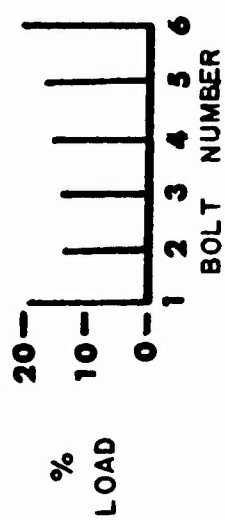


Figure 12a. Specimen 6: Distortional Energy Contour Plots for the Experimental Failure Load.



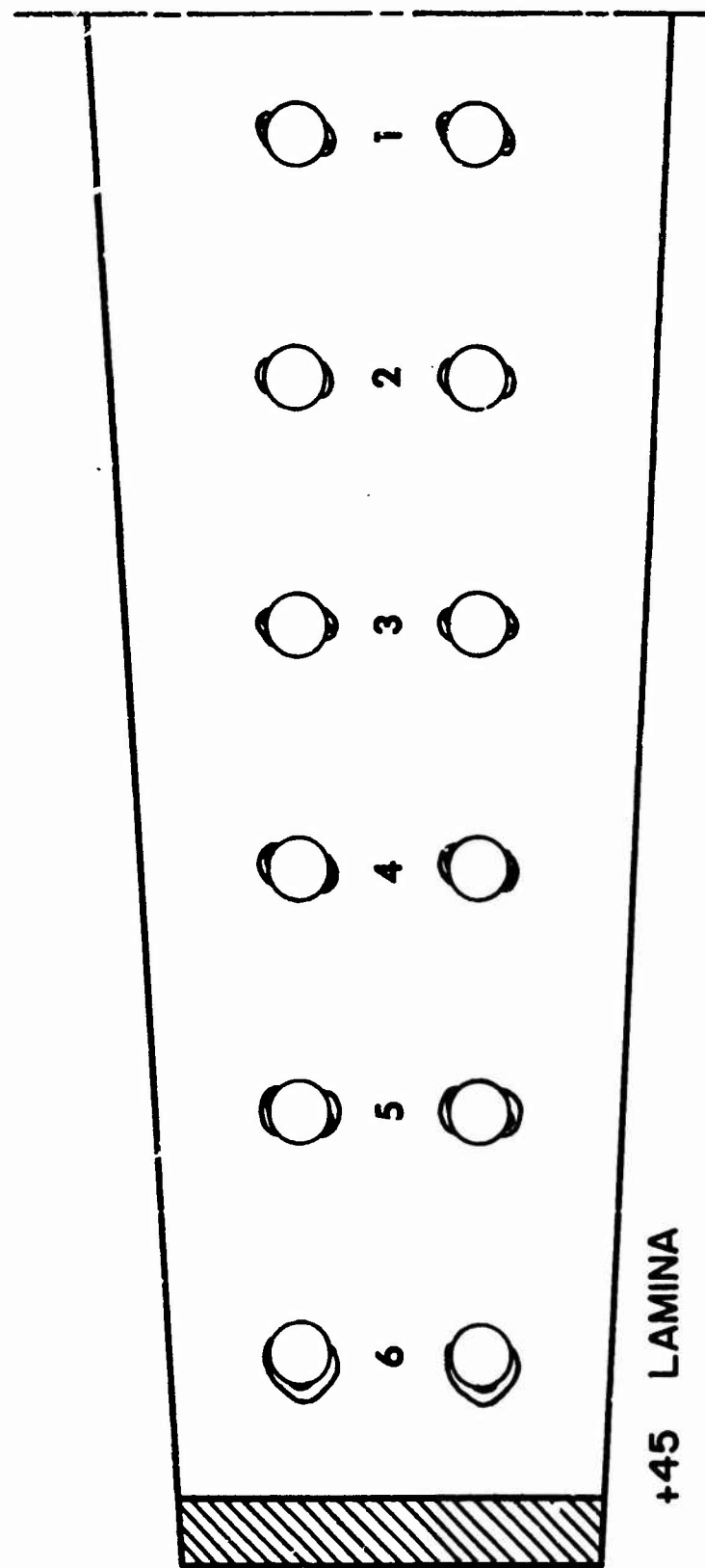


Figure 12b. Specimen 6: Distortional Energy Contour Plots for the Experimental Failure Load.

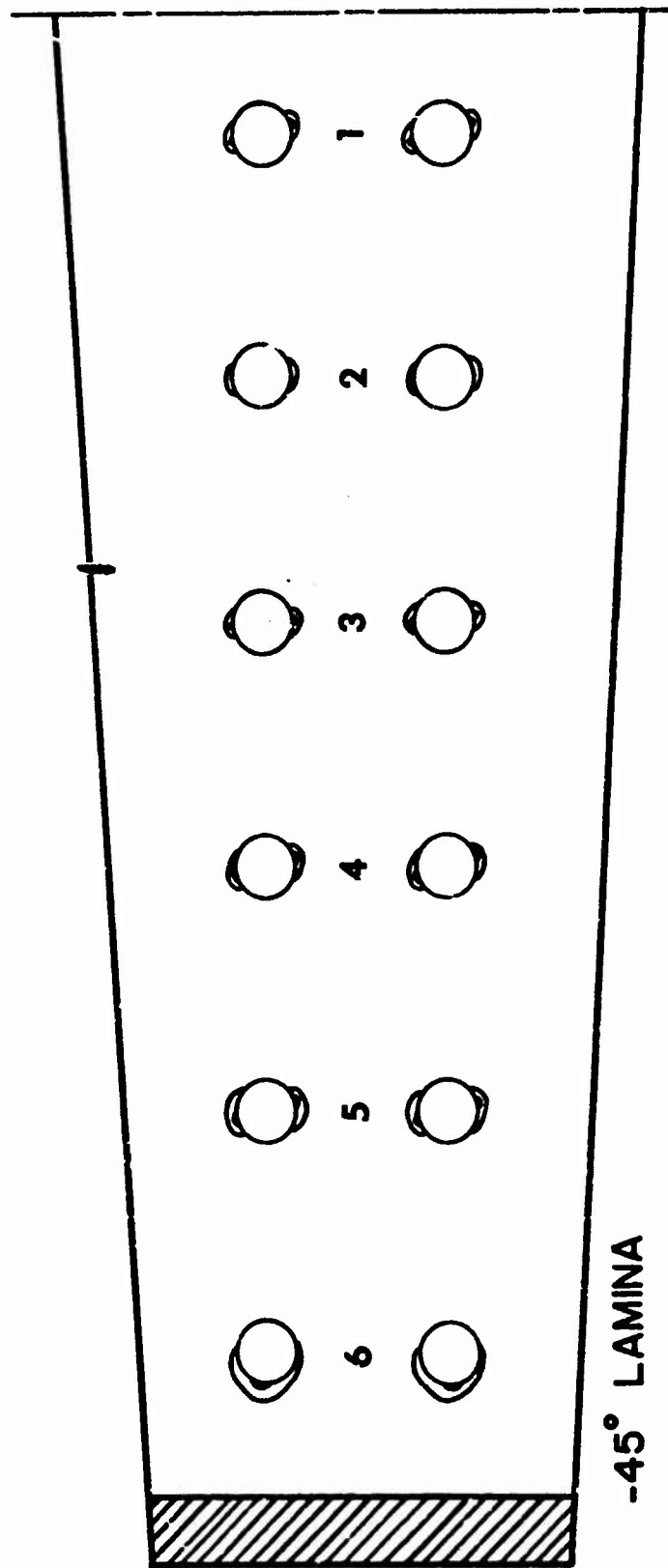
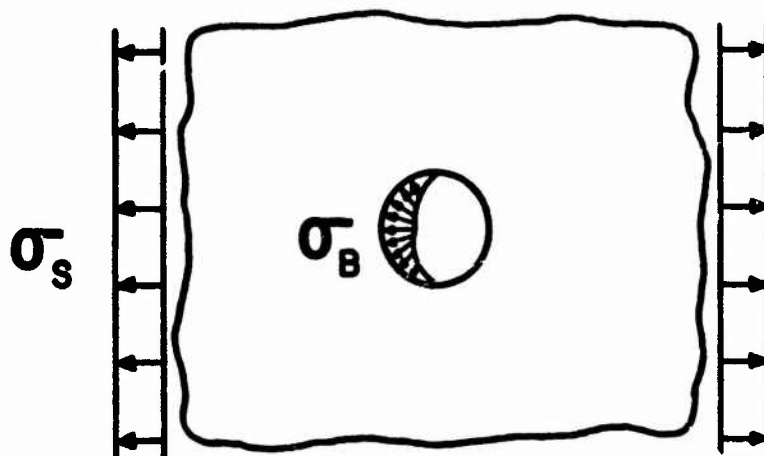
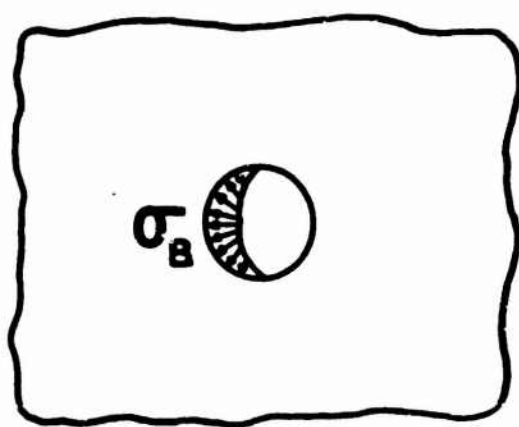


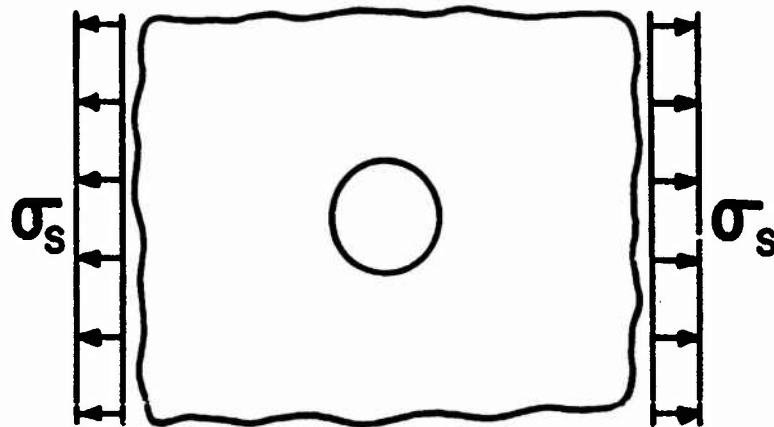
Figure 12c. Specimen 6: Distortional Energy Contour Plots for the Experimental Failure Load.



(a) Problem of interest



(b) Bolt load only



(c) Tension loading only

Figure 13. The Principle of Superposition Applied to an Infinite Bolt Bearing Model.



## CHAPTER IV

### OPTIMIZATION METHODS

#### 4.1 INTRODUCTION

A computer program for optimization using non-linear programming by pattern search (OPTIM), written by Martin Schussel [1] was used for this study. Some time was spent studying this program and a sample problem was run (torsion of an elliptic bar). The time spent in finding the predicted result of this problem yielded much insight into optimization techniques and the OPTIM program itself.

The bolt-bearing problem was analyzed, using OPTIM, which included variation of the ply orientations. This study involved a problem with six variables and four constraints. For a given load, the minimum weight dimensions and orientations were found. The results of this study are discussed in some detail.

#### 4.2 STRUCTURAL OPTIMIZATION

##### 4.2.1 Background

The structural optimization project consisted of finding the minimum weight design of a structure for which certain limitations were posed. The limitations or constraints can be of the following form:

- a) Geometric
  - Maximum overall dimensions of the structure
  - Maximum thickness, cross-section, length, width,  
etc. of an internal member
  - Maximum deflection of a member
  - Maximum rates of deflection

- b) Mechanical
- Yield criterion
  - Failure modes
  - Fatigue properties
  - Natural frequencies
  - Buckling loads

If the structure can be analytically solved for an internal stress state as a function of the external loads (given) and the dimensions of the piece (to be used as variables) then the problem becomes a mathematical one: Find the extreme values of a non-linear function of several variables, subject to one or several non-linear constraints. The function is usually the weight of the structure and the variables are its dimensions. The constraints can be in the form of equalities or inequalities. The equality constraints would generally concern a total dimension which is not fixed but is the sum of a number of internal dimensions. Inequality constraints are far more common, they usually insure that yield stresses, buckling loads, etc. are not exceeded.

#### 4.2.2 Variational Method

There are several methods of mathematically solving the problem, but non-linear programming is the only reliable one. Graphical methods have a very limited use as they can only be used in two-dimensional problems. Transformation into a series of linear problems by use of Taylor series expansions is tedious and inaccurate. The use of penalty functions transforms the problem into an unconstrained minimization. Lagrange multipliers are an example; the formulation of the problem with Lagrange multipliers is as follows:

Assume we want to minimize a weight function  $W(X_i)$  where  $X_i$  ( $i = 1, \dots, N$ ) are the variables. The constraints to be satisfied are

are  $F_j(x_i) = 0$  ( $j = 1, \dots, M$ ). The Lagrange multipliers ( $\lambda_j$ ) are added and we form an unconstrained objective function ( $P$ ) to be minimized.

$$P = W(x_i) + \lambda_j F_j(x_i)$$

Now setting the derivatives to zero will find the extrema:

$$\frac{\partial P}{\partial x_i} = 0 \quad i = 1, \dots, N$$

$$\frac{\partial P}{\partial \lambda_j} = 0 \quad j = 1, \dots, M$$

The problem now requires the solution of  $N + M$  simultaneous non-linear algebraic equations in  $N + M$  unknowns. Solutions are difficult to find and are not unique, so this method is useless for large, complicated problems.

#### 4.2.3 Non-linear Programming Methods

By far the most useful methods for solving non-linear optimization problems are searching techniques. There are many methods of search mentioned in the literature (pattern search, directed search, Fibonacci search, steepest ascent search), but basically they all consist of searching the domain of the variables until no further improvement can be found in the objective function.

Included in the Appendix of [2] are the listing and instructions for a pattern search optimization program (OPTIM) by Martin Schussel, Carnegie-Mellon University 1968. The program works in the following way: An objective function is defined:

$$P = \text{COST} + \sum A(K) (F(K))^2$$

where COST = weight function

$A(K)$  = penalty functions

$F(K)$  = constraints

COST and  $F(K)$  which are functions of the variables (  $X(I)$  ) are defined by the user of the program in a subroutine called CALC.

The program increases and decreases the variables and recalculates COST and  $F(K)$  until the improvement in the objective function is smaller percentagewise than  $10^{-5}$ .

The program is best suited to handle inequality constraints (less than or equal) which it handles in the following way: If the constraint becomes negative during the search it is neglected, but if it becomes positive it is multiplied by a penalty (some large number  $A(K)$ ). When the objective function is minimized the constraints will either approach zero or remain negative.

The application of optimization to design of structures using advanced fiber composite materials adds another facet to the problem. The orientations of the plies become variables as well as the dimensions. In some cases, the problem can be handled similarly to the above procedure with the orientations merely being additional variables. However, analytical equations for composite materials are difficult to derive and are usually not solvable in closed form. The bolt bearing problem was solved using empirical equations which relate the failure loads to the dimensions of the plate and ply orientations.

#### 4.3 TORSION OF AN ELLIPTIC BAR — VARIATIONAL EXAMPLE

The problem is to find the values of the major and minor axes of an elliptic bar for minimum weight for a given applied torsional moment  $M$ . The weight is proportional to the cross sectional area

$$A = \pi ab \tag{1}$$



We want to minimize  $A$ , subject to the constraint that the allowable shear stress is not exceeded at any point. The maximum stress is at  $y = b$ ,  $x = 0$  and is given by

$$\tau_{\max} = \frac{2M}{\pi ab^2} \quad (2)$$

If  $\tau_y$  is the yield stress, the constraint equation becomes

$$\begin{aligned} \frac{2M}{\pi ab^2} - \tau_y &\leq 0 \\ \text{or } 2M - \pi ab^2 \tau_y &\leq 0 \end{aligned} \quad (3)$$

The solution was then sought using Lagrange multipliers. The results were incorrect since two more constraints must be added. The first one is due to the fact that the stress formula is only correct if  $a$  is larger than  $b$ .

$$\begin{aligned} b &\leq a \\ b - a &\leq 0 \end{aligned} \quad (4)$$

We must also insure that  $b$  and  $a$  are positive for the answers to make sense. If we insure that  $b$  is positive, the first constraint allows  $a$  to be positive, thus the last necessary constraint is

$$b \geq 0 \quad (5)$$

We can now change the problem into an unconstrained minimization problem by the use of Lagrange multipliers and slack variables. The solution is to find an extreme value of a function  $F$  by variational methods where  $F$  is given by

$$\begin{aligned} F = \pi ab + \lambda_1 \left( \pi ab^2 - \frac{2M}{\tau_y} - \gamma^2 \right) \\ + \lambda_2 (a - b - \beta^2) + \lambda_3 (b - \delta^2) \end{aligned} \quad (6)$$

where  $\lambda_1, \lambda_2, \lambda_3$  are the Lagrange multipliers, and  $\gamma, \beta, \delta$  are the slack variables used in inequality constraints. The derivatives of (6) with respect to each variable yields the following set of equations

$$\begin{aligned}
 \frac{\partial F}{\partial a} &= 0 = \pi b + \lambda_1 \pi b^2 + \lambda_2 \\
 \frac{\partial F}{\partial b} &= 0 = \pi a + \lambda_1 2\pi ab - \lambda_2 + \lambda_3 \\
 \frac{\partial F}{\partial \lambda_1} &= 0 = \pi ab^2 - \frac{2M}{\tau_y} - \gamma^2 \\
 \frac{\partial F}{\partial \lambda_2} &= 0 = a - b - \beta^2 \\
 \frac{\partial F}{\partial \lambda_3} &= 0 = b - \delta^2 \\
 \frac{\partial F}{\partial \gamma} &= 0 = -2\gamma\lambda_1 \\
 \frac{\partial F}{\partial \beta} &= 0 = -2\beta\lambda_2 \\
 \frac{\partial F}{\partial \delta} &= 0 = -2\delta\lambda_3
 \end{aligned} \tag{7}$$

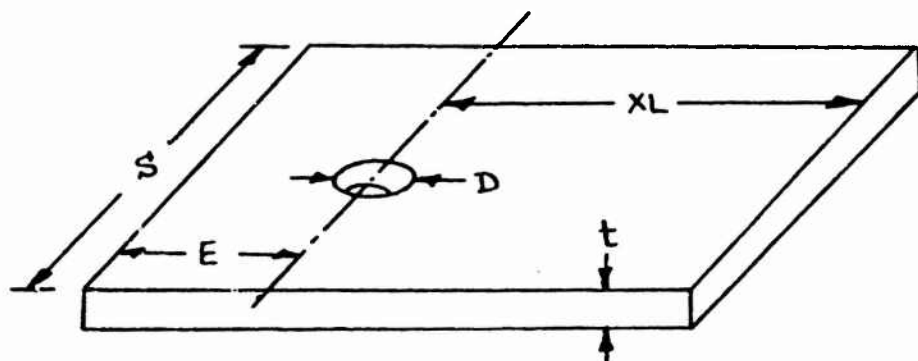
Since  $\beta=0$ , (4) gives  $a=b$  (circular section); Eqn (3) gives  $\pi ab^2 = \frac{2M}{\tau_y}$  for  $\gamma=0$ . Since  $a=b$ , we have  $\pi a^3 = \frac{2M}{\tau_y}$  and thus

$$a = [2M/\pi\tau_y]^{1/3} \tag{8}$$

The problem was also solved using the optimization computer program (OPTIM). First the problem was attempted using only the first constraint and a minimum was found with  $b$  about twice the size of  $a$ . This violated the condition that  $a$  be greater than or equal to  $b$  for the stress equation to apply. Next all three constraints were used and the minimum was found to agree with the analytical result (8). Thus we conclude that inclusion of all constraint relations is absolutely essential for success.

#### 4.4 BOLT BEARING PROBLEM - NON-LINEAR PROGRAMMING EXAMPLE

The problem consists of finding the minimum weight of plates loaded by bolted joints. The specimen appears as shown below:



$XL$  is a constant

$D$  is chosen as .375 in.

The weight of the specimen is:

$$W = \rho(XL + E)S t \quad (9)$$

The weight of the material which would be in the hole is included since it must be wasted. Empirical equations [3] for the three failure modes found in experiments are as follows:

*Tension*

$$P \leq .69 t(S-D)F^{tu} \quad (10)$$

where  $P$  = applied load and  $P$  cannot exceed the expression on the right.

The symbol  $F^{tu}$  is defined as:

$$F^{tu} = \frac{157L + \left(\frac{20.7N^2}{3M + N}\right)}{1 + .0538 \left[4 \frac{M}{L} - \left(\frac{M}{L}\right)^2\right]} \quad \text{for } \frac{M}{L} \leq 2 \quad (11)$$

$$F^{tu} = 129L + 27N - \frac{N(10N + 162L)}{3M + N} \quad \text{for } \frac{M}{L} > 2$$

where L = % 0° plies

M = % 90° plies

N = %  $\pm 45^\circ$  plies

The constraint for this failure mode is  $F(1) = P - .69 t(S-D)F^{tu}$ . If this quantity stays negative then P is below failure load. If it is near zero, failure in this mode is impending. The problem was treated from two different viewpoints.

First the orientation percentages L, M, and N were held constant and the dimensions for minimum weight of the specimen were found. The answer in this case yields the optimum dimensions for the orientations chosen.

The second way of treating the problem was to leave the orientations as variable. This way, both the dimensions and the orientations were optimized. The results showed a 20-30% improvement over the fixed orientations case. The orientations chosen were those of an experimental specimen which failed at  $P = 1020$  lb. The program gave a weight reduction for failure at the same load and with the same orientations.

The problem also included equations for failure in two other modes-shear out and bearing. These were the second and third constraints. The failure mode in a given problem is found by checking which of the three constraints is closest to zero. The constraints are

*Shear Out*

$$\begin{aligned} P &\leq 2tEF^{su} \\ F(2) &= P - 2tEF^{su} \end{aligned} \tag{12}$$

where

$$\begin{aligned} F^{su} &= 40N & N &\geq .23 \\ F^{su} &= 9.2 & N &< .23 \end{aligned}$$

### Bearing

$$P \leq \frac{Dt}{4} (1 + .45 \frac{D}{t}) F^{bu} \quad (13)$$

$$F(3) = P - \frac{Dt}{4} (1 + .45 \frac{D}{t}) F^{bu}$$

where

$$F^{bu} = L F^L + (M + N) F^M$$

$$\begin{aligned} \text{if } L \geq .25 & \quad F^L = 600, F^M = 30 \\ L \leq .25 & \quad F^L = 450, F^M = 80 \end{aligned}$$

The results of the program for fixed orientations are shown in Table I.

Choose L = 18.2%	(0° plies)
M = 9.1%	(90° plies)
N = 72.7%	(±45° plies)

The orientations in Table I were chosen because test data was available for a failed specimen. The specimen failed at 1000 lbs. and had the dimensions shown below:

P	THK	EDGE	SIDE	COST
1000	.056	.50	1.0	.044

The optimum dimensions for P = 1000 give COST = .039 (10% weight reduction).

Table II contains the results of the analysis for the case of using the orientations as variables. Surprisingly, the optimum orientations do not change for different loads.

The orientations were allowed to vary between .10 and .80 in the above procedure.

The results in Table III were found for variable orientations with the possibility of eliminating certain plies. There is some doubt of the applicability of the equations for less than 10% of any of the plies, but it is informative to see what will happen in this case.

The results for all three cases are plotted together for comparison in Fig 3.

#### 4.5 DISCUSSION

The OPTIM program has proven to be very effective in dealing with problems for which analytical equations can be derived. The elliptic bar and bolt bearing problems treated above are examples.

The bolt-bearing problem is unusual for composite materials in that analytical equations are available which allow us to optimize both the dimensions and the lamina orientations. The equations are empirical and therefore introduce doubt as to their accuracy. There also may be ranges of dimensions or orientation percentages in which they are not applicable.

Table I shows optimum dimensions for varying load with the ply orientations fixed. The case of  $P = 1000$  lb. shows a 10% weight reduction over the experimental specimen. The values of the constraints show this to be a simultaneous failure in tension and shear out. The cases of  $P$  (applied load) between 3000 lbs. and 15,000 lbs. show failure in all three modes simultaneously. There is no apparent pattern in the variation of optimum dimensions with load. The weight is seen to vary non-linearly with load as can be inferred from Fig 1.

If we allow the orientation percents to vary between 10% and 80% the optimum laminate will be found with respect to both dimensions and

orientations. OPTIM found values of five of the six variables which were optimum for all loads considered. Only the total thickness changed and it varied linearly with load. This situation forced the weight to vary linearly with load also as seen in Fig 2. For each load, the specimen exhibited failure in tension and shear out simultaneously with bearing failure not being a factor. The orientations chosen for each load were  $L = 73\%$ ,  $M = 17\%$ ,  $N = 10\%$ . The fact that  $N$  was brought to the minimum of its range led to the results in Table 3 where  $N$  was allowed to vary between 0% and 80%. The results are similar to those in Table 2 except that  $N$  goes to zero with  $L$  and  $M$  increasing proportionately. As noted, the equations may not apply for  $N$  less than 10%, but the results indicate that the  $\pm 45^\circ$  laminae are of little benefit in the bolt bearing specimen. The thickness and weight vary linearly with load as in the previous case. All three cases are plotted in Fig 3. The variable orientation case shows an improvement on the fixed case of between 30% and 100%, with the case for  $N = 0$  about 15% better still.

The results show a useful and convenient relationship for design. The designer is given the optimum orientations and side and edge distances and he merely chooses his thickness to suit the load which must be carried. The empirical nature of the equations suggests that experiments should be run to verify the derived results before putting them into use as a design criterion.

#### 4.6 REFERENCES

- [1] M. D. Schussel, "Optimization by Pattern Search Program Description and Usage Instructions", prepared for GE-580, Department of Mechanical Engineering, Carnegie-Mellon University (1968).
- [2] S. J. Marulis, "Optimization of Advanced Composite Plates", Report SM-71, Department of Mechanical Engineering, Carnegie-Mellon University, Pittsburgh, Pennsylvania (June 1971).
- [3] Advanced Composite Wing Structures, Boron-Epoxy Design Data - Volume II, Grumman Aerospace Corporation, 180-186, TR-AC-SM-ST-8085 (November 1969).



Table 1: Coupon Weights for Fixed Orientations

P (LOAD)	THICKNESS	EDGE DIST.	SIDE DIST.	F(1) TENSION FAILURE	F(2) SHEAR OUT FAILURE	F(3) BEARING FAILURE	COST (WEIGHT)
1000	.014	1.31	3.42	-.0061	-.0093	-.1485.	.039
3000	.048	1.07	2.87	-.0024	-.00073	-.144	.115
7500	.374	3.45	1.18	-.037	-.028	-.169	.342
10,000	.559	.31	1.09	-.051	-.20	-.210	.473
15,000	.916	.28	1.03	-.069	-.58	-.484	.729

Table 2: Coupon Weights for Variable Orientations

P (LOAD)	THICK NESS	EDGE DIST.	SIDE DIST.	F(1) TENSION FAILURE	F(2) SHEAR OUT FAILURE	F(3) BEARING FAILURE	L 0°	M 90°	N ±45°	COST (WEIGHT)
1000	.016	3.32	1.52	-.0098	-.0068	-6602.	.72	.18	.10	.025
3000	.050	3.34	1.49	-.0013	-.019	-6207.	.74	.16	.10	.075
5000	.082	3.29	1.53	-.013	-.028	-5547.	.73	.17	.10	.124
7500	.125	3.31	1.51	-.0023	-.010	-4808.	.73	.17	.10	.186
10,000	.167	3.30	1.49	-.061	-.018	-5685.	.72	.18	.10	.251
15,000	.243	3.35	1.50	-.0033	-.071	-2350.	.74	.16	.10	.373

Table 3: Coupon Weights Allowing Orientations to be Eliminated

P (LOAD)	THICKNESS	EDGE DIST.	SIDE DIST.	F(1) TENSION FAILURE	F(2) SHEAR OUT FAILURE	F(3) BEARING FAILURE	L %°	M %90	N %±45°	COST (WEIGHT)
1000	.015	3.60	1.32	-.00084	.000	-7060	.77	.23	.00	.0207
3000	.049	3.34	1.23	-.0024	-.0011	-6846	.80	.20	.00	.0607
5000	.072	3.78	1.39	-.042	-.011	-5841	.79	.21	.00	.1014
7500	.109	3.71	1.32	-.039	-.0019	-5143	.80	.20	.00	.1506
10,000	.143	3.82	1.41	-.016	-.018	-4196	.80	.20	.00	.2008
15,000	.217	3.73	1.34	-.056	-.0021	-6032	.80	.20	.00	.3073

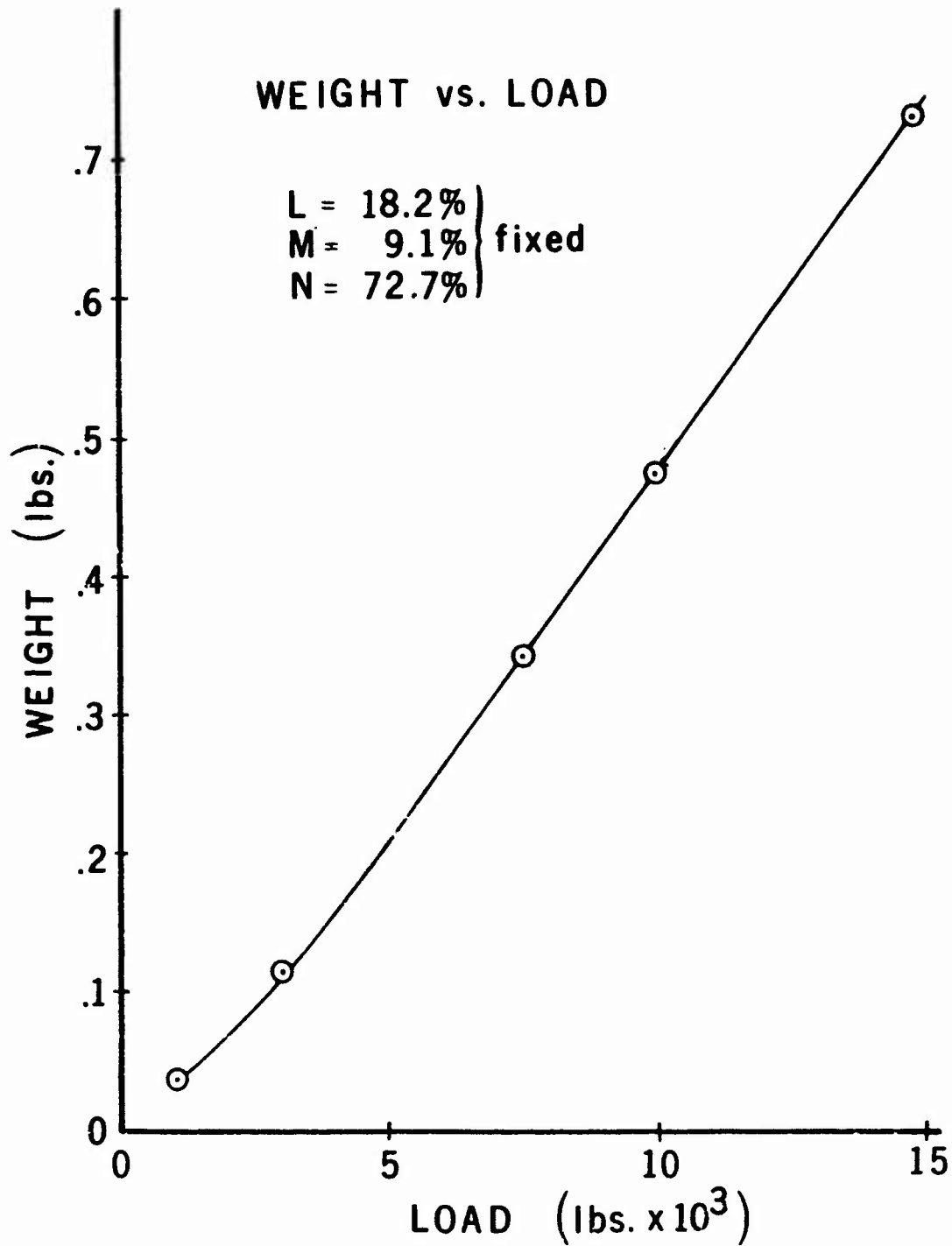


FIGURE 1: WEIGHT VS. LOAD FOR FIXED ORIENTATIONS

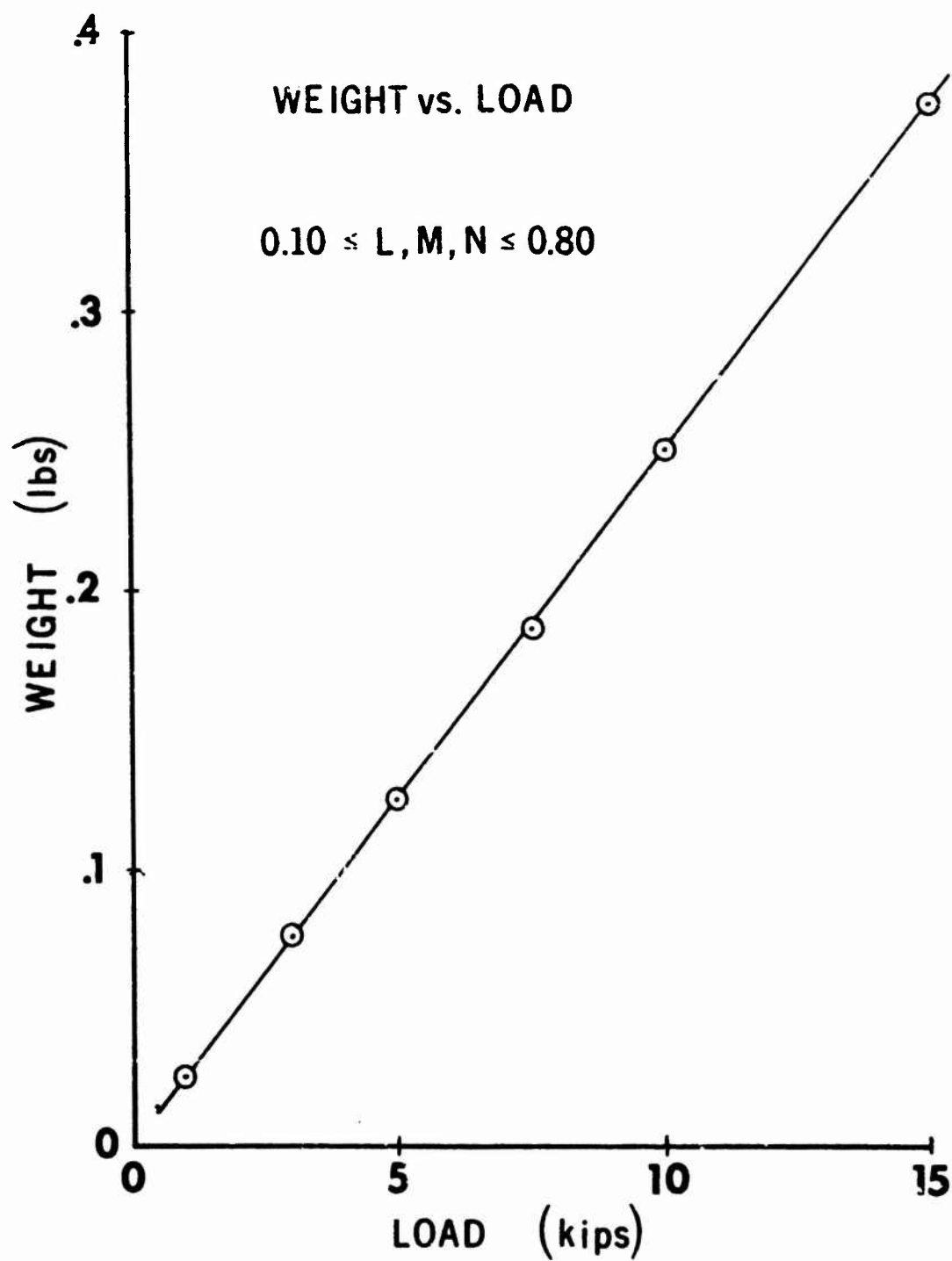


FIGURE 2: WEIGHT VS. LOAD FOR VARIABLE ORIENTATIONS

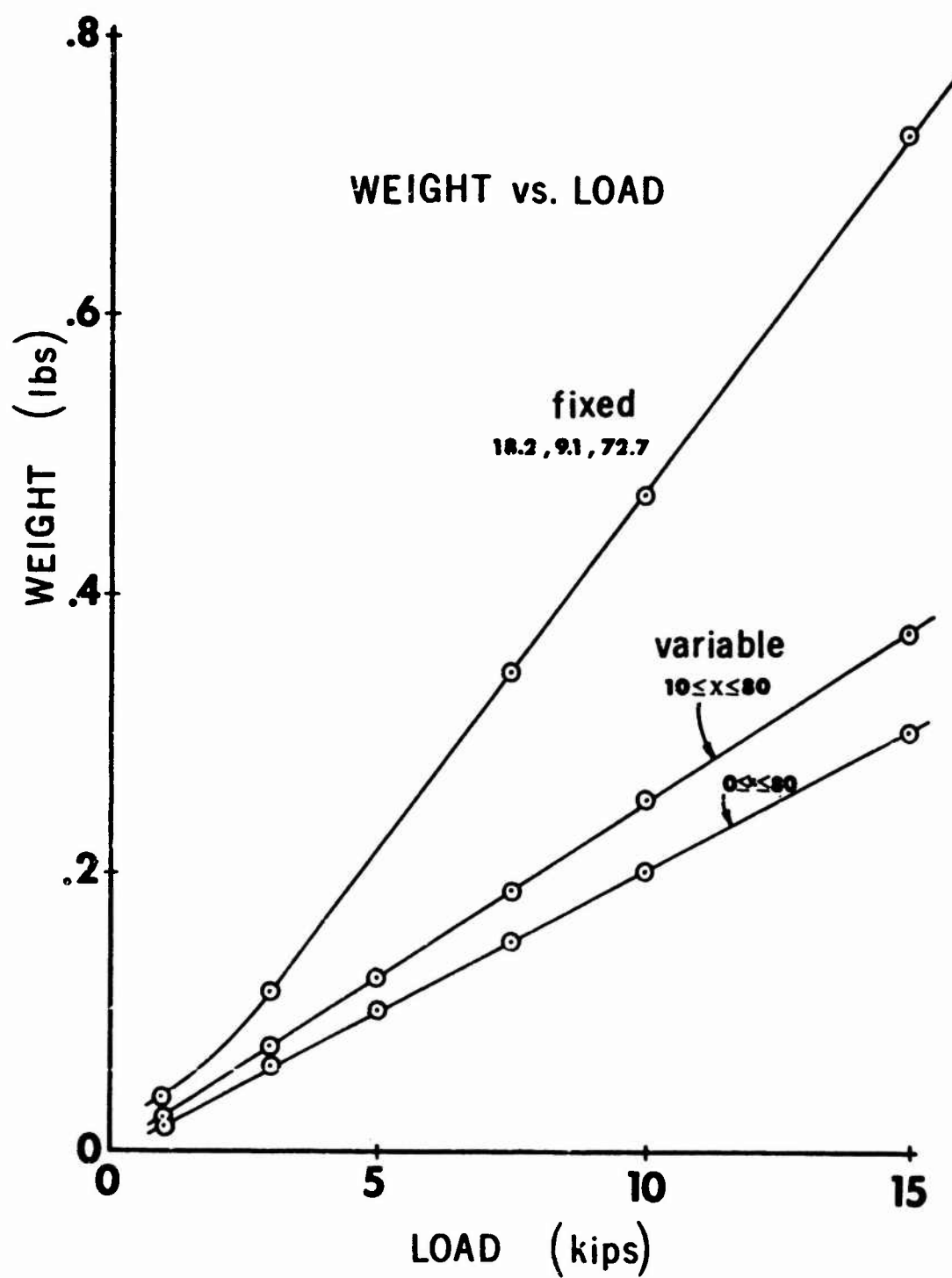


FIGURE 3: SUMMARY OF WEIGHT VS. LOAD RESULTS

## CHAPTER V

### BOUNDARY-INTEGRAL EQUATION SOLUTION METHODS

#### 5.1 TWO DIMENSIONAL ISOTROPIC BOUNDARY-INTEGRAL EQUATION METHOD

##### 5.1.1 Introduction

The boundary-integral equation method is a new tool for the solution of many problems in solid mechanics. The method has significant advantages over the finite element method. Numerical approximations are not made over the field but over the surface, thereby increasing accuracy. The dimension of the problem is reduced by one, allowing many problems too large for today's computers to be solved. Both of these features permit the analyst to obtain highly refined data in the vicinity of stress concentrations such as near cracks and notches.

Important to the user of the boundary-integral equation (BIE) method, is the ease of data preparation and the rapidity of solution. The BIE method utilizes a numerical solution of a boundary constraint equation. This equation relates all of the surface displacements to all of the surface tractions. The analyst specifies how he wishes to subdivide the surface and specifies the boundary data; all well-posed problems are acceptable including mixed-mixed problems. The geometry is completely general and may be multiply-connected. Once the surface solution is found the stresses may be generated at *any* points that the analyst desires on the interior of the region.

The BIE method has been widely adapted to many problems in solid mechanics, as can be seen by the literature [1-8]. The purpose for presenting it in this report is twofold. First, the tool is being

developed by the CMU team for two dimensional, anisotropic problems for use in several on-going research efforts. Second, it is desirable to make the method available to the widest possible group of users. Listings of both the isotropic and anisotropic computer programs are therefore contained in this Chapter.

### 5.1.2 Review of the Isotropic Boundary-Integral Equation Method

Two elements are required for the development of the boundary constraint equation of the BIE method. The first is a reciprocal relation between two solution states (Betti's reciprocal work theorem); the second is a fundamental solution or influence function (Kelvin's problem of a point load in an infinite body). The development herein follows that used in classical potential theory (see, for example, [9-13]).

The solution to Kelvin's problem consists of displacement vectors in each of the  $x_j$  directions due to concentrated loads applied in the  $x_i$  directions. These solutions are denoted by the displacement tensor  $U_{ij}$ ; the appropriate forms can be found in the literature [1-10]. In two dimensional, isotropic, elastostatics this tensor is

$$U_{ij}(P,Q) = - [\ln(1/r(P,Q)) (3-4\nu)\delta_{ij} + r_{,i}r_{,j}]/8\pi\mu(1-\nu) \quad (1)$$

In (1) the distance between the point of load application  $P(x)$  and the field point  $Q(x)$  is denoted  $r(P,Q)$ ;  $\mu$  and  $\nu$  are the shear modulus and Poisson's ratio. The derivative of  $r(P,Q)$  in the  $x_i$  direction is denoted

$$r_{,i} = \frac{\partial r}{\partial x_i|_Q} = \frac{x_i|_Q - x_i|_P}{r(P,Q)} \quad (2)$$

It is easily shown that (1) satisfies Navier's equation of equilibrium

$$(1/1-2\nu)u_{i,ij} + u_{j,ii} \quad (3)$$



A second tensor is required for the use of the reciprocal work theorem: the tractions corresponding to the  $U_{ij}$  on the physical surface  $\partial R$  of the body. These tractions,  $T_{ij}$ , are obtained by using Hooke's law and the definition of the traction vector

$$t_i = \sigma_{ij} n_j = \mu [(2\nu/(1-2\nu)) u_{k,k} \delta_{ij} + u_{i,j} + u_{j,i}] n_j \quad (4)$$

Utilizing (1) and (4) the traction tensor  $T_{ij}$  is found

$$T_{ij} = \{ \partial r / \partial n [(1-2\nu) \delta_{ij} + 2r_{,i} r_{,j}] + (1-2\nu) (n_i r_{,j} - n_j r_{,i}) \} / 4\pi(1-\nu)r(P,Q) \quad (5)$$

After some amount of manipulation of the reciprocal work theorem and letting  $P, Q$  be boundary points ( $P$  not at a corner) the following boundary constraint equation can be found

$$u_i(P)/2 + \int_{\partial R} u_j(Q) T_{ij}(P,Q) dS(Q) = \int_{\partial R} t_j(Q) U_{ij}(P,Q) dS(Q) \quad (6)$$

In (6)  $u_i, t_i$  are the displacements and tractions on the physical surface  $\partial R$  for the problem to be solved.

The numerical solution to (6) is obtained by discretizing the boundary and boundary data in some suitable fashion. Presently the displacements,  $u_i$ , and tractions,  $t_i$ , are taken as piecewise constant over each of  $N$  boundary segments. Work is well underway to use linear variations. The boundary segments are assumed to be flat in the programs used by this investigator. This allows for a completely general computer program for arbitrary surface shapes. When the approximations are made (6) becomes

$$u_i(P_m)/2 + \sum_{n=1}^N u_j(Q_n) \int_{\partial R_n} T_{ij}(P_m, Q) dS(Q) = \sum_{n=1}^N t_j(Q_n) \int_{\partial R_n} U_{ij}(P_m, Q) dS(Q) \quad (7)$$

Eq. (7) can be written in matrix form as

$$(1/2[I] + [\Delta T]) \{u\} = [\Delta U] \{t\} \quad (8)$$

where  $[I]$  is the identity matrix;  $[\Delta T]$  and  $[\Delta U]$  are coefficient matrices from the integrations in (7): These integrals are calculated analytically in the program by specifying the coordinates of the ends of the boundary segments.

When the boundary data for a well-posed problem are specified then  $2N$  quantities in (8) are known and  $2N$  quantities are unknown. Standard reduction schemes are employed to solve for the unknowns. After the entirety of the surface data is formed the interior stresses at any selected points are found by the quadrature relation

$$\sigma_{ij}(p) = \sum_{n=1}^N u_k(Q_n) \Delta S_{kij}(p, Q_n) - \sum_{n=1}^N t_k(Q_n) \Delta D_{kij}(p, Q_n) \quad (9)$$

The tensors  $\Delta S_{kij}$  and  $\Delta D_{kij}$  are calculated as indicated in [7]. A procedure for calculating the stress tensor at the surface is accomplished using surface displacements and tractions as discussed in [8].

### 5.1.3 Use of the Isotropic Computer Program

The isotropic version of the program is limited to linear, isotropic, homogeneous, elastic problems with known material constants  $\mu$  (or  $G$ , shear modulus), defined as FMU in the program, and  $\nu$ , defined as POISN, or PR, in the program. The user has available four operating modes for the program:

*Boundary Solution:* This capability is the first step always for each problem as it solves (8) for all unknown boundary data in terms of specified boundary conditions and geometry. The entire set of boundary data may be output on punched cards (see next section).

*Interior Solution:* Upon completion of the boundary solution the analyst may request stress solutions, using (9), at as many interior points he desires by specifying their number and location.

*Boundary Solution:* The boundary stress solution is based on the same finite difference result discussed in the Appendix of [8]. The solution is obtained at a specified boundary segment from the known or calculated surface tractions and the calculated tangential derivative of displacements. The means for calculating the tangential derivative is discussed in greater detail in the next section.

*Restart:* By reading the entire set of boundary data the program may solve directly for interior or boundary stresses.

#### 5.1.3.1 Dimension Statements

The current version of the program (See Section 5.1.5) admits up to two degrees of symmetry of geometry and boundary conditions. The program is limited to a total of 80 boundary segments (320 with symmetry). To increase the size of the program change the following cards,

COMMON / ARRAY1 / ...

COMMON / ARRAY2 / ...

in the various routines; also the following sequence numbered cards should be changed

10060            20035

10065

10075            50005

15050

15200

The program is limited to 200 interior solution points, COMMON / ARRAY3 / ..., and to 50 surface points, COMMON / ARRAY4 / .

An 1108 assembler language routine for calculating time is attached for 1108 users. Other users must supply a similar subroutine to obtain a time-breakdown chart for each solution; if not available, insert a *dummy* subroutine, SUBROUTINE TIME (T).

#### 5.1.3.2 Definition of Key Parameters, Matrices

The key parameters are described in cards 15060 - 15115, in SETUP. These parameters govern geometry (NSEG, NSYM, NNOD), execution options (IPUNCH, ISTRS, IBDY), and particular stress solutions (NPT, NBDYP). The first card read is a TITLE card followed by the control numbers, read by cards 15120 and 15125.

The temporary array NODE (I,J) stores the two node numbers associated with each segment number and is read by card 15130. The temporary array XYZM (I,J) reads in the  $x_1$ ,  $x_2$  coordinates of each of the nodes by card 15135.<sup>1</sup> The material constants FMU, POISN are then read by card 15140.

At this time the program merges the geometric information to form the permanent geometric array XYZ (Segment Number, Node Number, Coordinate Number). If NPT  $\neq$  0, the coordinates of the interior stress points are read in by card 15225. If NBDYP  $\neq$  0, three segment numbers are read by card 15240. The three numbers in NBDY (I,J) have the following meanings:

NBDY (Segment No., 1) = Segment number for which stress  
calculations is to be done.

---

<sup>1</sup> Only the geometry for the basic symmetric part is read in. If NSYM  $\neq$  0, the program assumes one degree of symmetry ( $y=0$  axis), or two degrees of symmetry ( $y=0$  axis, then  $x=0$  axis) according to NSYM.

NBDY (Segment No., 2) = Segment number for the "rear" point  
in calculating  $\Delta U/\Delta S$ .

NBDY (Segment No., 3) = Segment number for the "forward"  
point in calculating  $\Delta U/\Delta S$ .

**NOTE:** The sequence of numbers in NODE and NBDY is the "rear" number,  
then the "forward" number. The positive - s direction is  
*always* taken such that the material is always on the left.

#### 5.1.3.3 Boundary Conditions

The current version of the program uses a NAMELIST read  
(Fortran IV) statement. The procedure is to precede and close the  
block of boundary data with control cards in the following way

—\$ BDYCON

DATA

—\$ END

See standard references for formats for the data block.

**NOTE:** When NSYM = 1,0 the solutions admit a rigid body motion in  
the unconstrained direction(s) (x,y). A displacement freedom  
is fixed by letting LDC for that freedom be set to "2".

All boundary conditions are *initialized to zero* and LDC is initialized  
to "1". All x-direction data is stored, then y-direction data is stored:

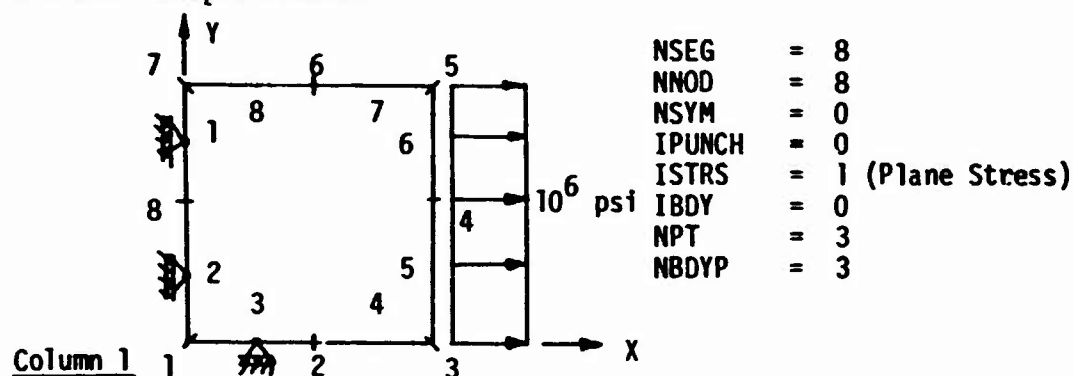
$$\left\{ \begin{array}{l} t_x \\ t_y \end{array} \right\} \begin{array}{l} 1, \text{ NSEG} \\ \text{NSEG} + 1, 2 * \text{NSEG} \end{array} \right\} = \left\{ \text{TCON} \right\}$$

etc. LDC = 1, means traction boundary conditions for the given segment and direction. LDC = 2 means displacement boundary condition for given segment and direction.

#### 5.1.3.4 Input Cards:

Information	No. Cards
Title	1
Control parameters	1
NODE (NSEG,2)	1 + (NSEG/12)
XYZM (NNOD,2)	1 + (NNOD/8)
FMU, POISN	1
Boundary Conditions	?
PITN (NPT,2)	1 + (NPT/8)
NBDY (NBDYP,3)	1 + (NBDYP/8)

#### 5.1.3.5 Example Problem



NSEG = 8  
 NNOD = 8  
 NSYM = 0  
 IPUNCH = 0  
 ISTRS = 1 (Plane Stress)  
 IBDY = 0  
 NPT = 3  
 NBDYP = 3

Column 1

NODE: --1--2--2--3--3--4--4--5--5--6--6--7--7--8--8--1  
 XYZM: -0000-2000-0000-1000-0000-0000-1000-0000-2000-0000-2000-1000-  
 -2000-2000-1000-2000  
 FMU, POISN: ----.1153846E+08----.30000  
 PITN: -0500-1000-1000-1000-0750-0750  
 NBDY: --7--7--8--7--6--7--6--5--6  
 NAMELIST:  
 -TCON(5) = 1.0E+6, TCON(6) = 1.0E+6,  
 -LDC(1) = 2, LDC(2) = 2, LDC(11) = 2,

#### 5.1.4 References

- [1] F. J. Rizzo, An Integral Approach to Boundary Value Problems of Classical Elastostatics, *Quarterly of Applied Mathematics* 25, (1967) 83-95.
- [2] T. A. Cruse and F. J. Rizzo, A Direct Formulation and Numerical Solution of the General Transient Elastodynamic Problem - I, *Journal of Mathematical Analysis and Applications* 22, (1968) 244-259.
- [3] T. A. Cruse, A Direct Formulation and Numerical Solution of the General Transient Elastodynamic Problem - II, *Journal of Mathematical Analysis and Applications* 22, (1968) 341-355.
- [4] T. A. Cruse, Numerical Solutions in Three Dimensional Elastostatics, *International Journal of Solids Structures* 5, (1969) 1259-1274.
- [5] T. A. Cruse and W. Van Buren, Three Dimensional Elastic Stress Analysis of a Fracture Specimen with an Edge Crack, *International Journal of Fracture Mechanics* 6 (1970).
- [6] T. A. Cruse and J. L. Swedlow, A Direct Method for Solving Three-Dimensional Problems of Elasto-Plastic Flow, Report SM-43, Department of Mechanical Engineering, Carnegie-Mellon University (1970).
- [7] F. J. Rizzo and D. J. Shippy, A Formulation and Solution Procedure for the General Non-Homogeneous Elastic Inclusion Problem, *International Journal of Solids Structures* 4, (1968) 1161-1179.
- [8] F. J. Rizzo and D. J. Shippy, A Method for Stress Determination in Plane Anisotropic Elastic Bodies, *Journal of Composite Materials*, 4, (1970) 36-61.
- [9] W. V. Lovitt, *Linear Integral Equations*, Dover (1950).
- [10] O. D. Kellogg, *Foundations of Potential Theory*, Dover (1953).
- [11] V. D. Kupradze, *Potential Methods in the Theory of Elasticity*, D. Davey & Co. (1965).
- [12] S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, Pergamon Press (1965).
- [13] C. E. Pearson, *Theoretical Elasticity*, Ch. VI, Harvard University Press (1959).

### 5.1.5 Listing for Isotropic Boundary-Integral Equation Computer Program

C		20*10000
C	MAIN PROGRAM -- INITIALIZES DATA - CALLS SUBROUTINES	20*10005
C		20*10010
	COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), TCON(200), LDC(200)	20*10015
	COMMON / ARRAY2 / BVAL(200)	20*10020
	COMMON / MATCON / FMU, P ISN, P1, P1, P2, P3, P4, P5	20*10025
	COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NBDYP	20*10030
	COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, IRDY	20*10035
	COMMON / TIMERS / T (10)	20*10040
C		20*10045
C	THE DIMENSIONS OF THE FOLLOWING ARRAYS ARE PROBLEM DEPENDENT	20*10050
C		20*10055
	DIMENSION C(160,160)	20*10060
	DOUBLE PRECISION RHS(160)	20*10065
	P1 = 3.14159265	20*10070
05	CONTINUE	20*10075
	DO 10 I = 1,200	20*10080
	UCON(I) = 0.	20*10085
	TCON(I) = 0.	20*10090
10	BVAL(I) = 0.	20*10095
	CALC TIME ( T(1) )	20*10100
	DO 20 I = 2,10	20*10105
20	T(I) = 0.	20*10110
	CALC SETUP	20*10115
	IF (IBDY.NE.0) GO TO 30	20*10120
	CALC BVSOLU (C, RHS)	20*10125
30	CALC INSOLU (C)	20*10130
	CALL BDYSTR (C)	20*10135
C		20*10140
C	CALCULATE TIME CHART	20*10145
C		20*10150
	T(2) = (T(2)-T(1))*10**(-3)	20*10155
	T(4) = (T(4)-T(3))*10**(-3)	20*10160
	T(6) = (T(6)-T(5))*10**(-3)	20*10165
	T(8) = (T(8)-T(7))*10**(-3)	20*10170
	T(10) = (T(10)-T(9))*10**(-3)	20*10175
	WRITE (6,2000) TITL	20*10180
	WRITE (6,2100)	20*10185
	WRITE (6,2200) T(2), T(4), T(6), T(8), T(10)	20*10190
	GO TO 05	20*10195
	STOP	20*10200
1000	FORMAT ( 16A5)	20*10205
2000	FORMAT (1H1, 16A5)	20*10210
2100	FORMAT ( 21H TIME BREAKDOWN CHART //)	20*10215
2200	FORMAT ( 5X 15HTIME FOR SETUP F12.7, 2X 7HSECONDS //	20*10220
1	5X 15HTIME FOR DELINT F12.7, 2X 7HSECONDS //	20*10225
2	5X 15HTIME FOR SOLVER F12.7, 2X 7HSECONDS //	20*10230
3	5X 15HTIME FOR INSOLU F12.7, 2X 7HSECONDS //	20*10235
4	5X 15HTIME FOR BDYSOL F12.7, 2X 7HSECONDS)	20*10240
	END	20*10245



SUBROUTINE SETUP	20*15000
COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), TCON(200), LDC(200)	20*15005
COMMON / ARRAY2 / BVAL(200)	20*15010
COMMON / ARRAY3 / PTIN(100,2)	20*15015
COMMON / ARRAY4 / NHDY(50,3)	20*15020
COMMON / MATCON / FMU, POISN, PI, P1, P2, P3, P4, P5	20*15025
COMMON / TIMERS / TIM( 6)	20*15030
COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NBDYP	20*15035
COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, IBDY	20*15040
NAMLIST / HDYCON / UCON, TCON, LDC	20*15045
DIMENSION NODE(100,2), XYZM(100,2)	20*15050
EQUVALENCE (NODE, LDC), (XYZM, UCON)	20*15055
C	20*15060
C NSFG = NUMBER OF SEGMENTS ON THE BOUNDARY	20*15065
C NSYM = NUMBER OF DEGREES OF SYMMETRY STARTING WITH Y, THEN X	20*15070
C NNOD = NUMBER OF BOUNDARY NODES CONNECTING BOUNDARY SEGMENTS	20*15075
C IPUNCH = 1 -- THE BOUNDARY SOLUTION WILL BE PUNCHED OUT	20*15080
C ISTRS = 0, PLSTRN -- ISTRS = 1, PLSTRS	20*15085
C IF IBDY.EQ.0 --- BOUNDARY DATA STORED IN COMMON	20*15090
C IF IBDY.NE.0 --- BOUNDARY DATA READ IN FROM CARDS ADDED TO END	20*15095
C OF THE DATA DECK	20*15100
C NPT = NUMBER OF INTERIOR SOLUTION POINTS FOR STRESS SOLUTION	20*15105
C NBDYP = NUMBER OF BOUNDARY POINTS FOR STRESS SOLUTION	20*15110
C	20*15115
READ (5,1000) TITL	20*15120
READ (5,1100) NSEG, NSYM, NNOD, IPUNCH, ISTRS, IBDY, NPT, NBDYP	20*15125
READ (5,1200) ((NODE(I,J),J=1,2),I=1,NSFG)	20*15130
READ (5,1300) ((XYZM(I,J),J=1,2),I=1,NNOD)	20*15135
READ (5,1400) FMU, POISN	20*15140
WRITE (6,2000) TITL	20*15145
WRITE (6,2100) NSEG, NSYM, NNOD, IPUNCH, ISTRS, IBDY, NPT, NBDYP	20*15150
WRITE (6,2200) ((NODE(I,J),J=1,2),I=1,NSFG)	20*15155
WRITE (6,2300) ((XYZM(I,J),J=1,2),I=1,NNOD)	20*15160
WRITE (6,2400) FMU, POISN	20*15165
NSIZE = 2 * NSEG	20*15170
DO 10 I = 1,NSFG	20*15175
DO 10 J = 1,2	20*15180
DO 10 K = 1,2	20*15185
N = NODE(I,J)	20*15190
10 XYZ(I,J,K) = XYZM(N,K)	20*15195
DO 20 I = 1,200	20*15200
UCON(I) = 0.	20*15205
20 LDC(I) = 1	20*15210
READ (5,80YCON)	20*15215
IF (NPT.EQ.0) GO TO 30	20*15220
READ (5,1500) ((PTIN(I,J),J=1,2),I=1,NPT)	20*15225
WRITE (6,2500) ((PTIN(I,J),J=1,2),I=1,NPT)	20*15230
30 IF (NBDYP.EQ.0) GO TO 40	20*15235
READ (5,1600) ((NBDY(I,J),J=1,3),I=1,NBDYP)	20*15240
WRITE (6,2600) ((NBDY(I,J),J=1,3),I=1,NBDYP)	20*15245
40 CONTINUE	20*15250
NFAC = 2**NSYM	20*15255
IF (NSYM.EQ.0) NFAC = 1	20*15260
NTOTAL = NSEG * NFAC	20*15265
C	20*15270
C CALCULATE NEEDED MATERIAL CONSTANTS 170	20*15275
C	20*15280

IF (ISTR5.EQ.1) POISN = POISN/ (1.+POISN)	20*152A5
P1 = 1./ (4.*PI*FNU*(1.-POISN))	20*15290
P2 = 3.-4.*POISN	20*15295
P3 = 1./ (4.*PI*(1.-POISN))	20*15300
P4 = 1.-2.*POISN	20*15305
CALI TIME ( TIM(2) )	20*15310
RETURN	20*15315
1000 FORMAT (16A5)	20*15320
1100 FORMAT (10I5)	20*15325
C	20*15330
C ***** CAUTION***** FORMATS PROBLEM DEPENDENT ***** CAUTION *****	20*15335
C	20*15340
1200 FORMAT (24I3)	20*15345
1300 FORMAT (16F5.3)	20*15350
1400 FORMAT (E15.7, F10.5)	20*15355
1500 FORMAT (16F5.3)	20*15360
1600 FORMAT (24I3)	20*15365
2000 FORMAT (1H1, 10X, 16A5)	20*15370
2100 FORMAT (// 10I5)	20*15375
2200 FORMAT (// 4(3X 2I3))	20*15380
2300 FORMAT (// 4(3X 2F10.6))	20*15385
2400 FORMAT (// 5X E15.7, F10.5)	20*15390
2500 FORMAT (// 4(3X 2F10.6))	20*15395
2600 FORMAT (// 6(3X 3I3))	20*15400
END	20*15405

SUBROUTINE RVLSOLU (C, RHS)	20*20000
COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), TCON(200), LDC(200)	20*20005
COMMON / ARRAY2 / BVAL(200)	20*20010
COMMON / MATCON / FMU, POISN, P1, P1, P2, P3, P4, P5	20*20015
COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NBDYP	20*20020
COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, IBDY	20*20025
COMMON / TIMEKS / TIM (10)	20*20030
DIMENSION C(200), PXYZ(2), C(NSIZE,NSIZE)	20*20035
EQUIVALENCE (A, UCON)	20*20040
DOUBLE PRECISION RHS(NSIZE)	20*20045
NMAX = 2 * NSEG	20*20050
WRITE (6,2000) TITL	20*20055
IF (ISTRS.EQ.0) WRITE (6,20050)	20*20060
IF (ISTRS.EQ.1) WRITE (6,20060)	20*20065
WRITE (6,2100)	20*20070
C	20*20075
C WRITE THE STARTING BOUNDARY CONDITIONS	20*20080
C	20*20085
DO 10 I = 1, NSEG	20*20090
J = I + NSEG	20*20095
DO 15 N = 1, 2	20*20100
15 PXYZ(N) = (XYZ(I,1,N) + XYZ(I,2,N))/2.	20*20105
10 WRITE (6,2200) I, UCON(I), UCON(J), TCON(I), TCON(J),	20*20110
1 LDC(I), LDC(J), PXYZ(1), PXYZ(2)	20*20115
DO 20 I = 1, NMAX	20*20120
RHS(I) = 0.000	20*20125
IF (LDC(I).EQ.1) GO TO 30	20*20130
BVAL(I) = FMU * UCON(I)	20*20135
GO TO 20	20*20140
30 BVAL(I) = TCON(I)	20*20145
20 CONTINUE	20*20150
C	20*20155
C CALCULATE DELU, DELT, RHS	20*20160
C	20*20165
CALI TIME ( TIM(3) )	20*20170
CALI DELINT (C, RHS)	20*20175
CALI TIME ( TIM(4) )	20*20180
WRITE (6,3000) ((C(I,J), J=1, NSIZE), I=1, NSIZE)	20*20185
C	20*20190
C WRITE RIGHT HAND SIDE VECTOR	20*20195
C	20*20200
WRITE (6,2300) TITL	20*20205
DO 40 I = 1, NSEG	20*20210
J = I + NSEG	20*20215
40 WRITE (6,2400) I, RHS(I), RHS(J)	20*20220
C	20*20225
C SOLVE SYSTEM OF EQUATIONS	20*20230
C	20*20235
CALI TIME ( TIM(5) )	20*20240
CALI SOLVER (NMAX, RHS, A, C)	20*20245
CALI TIME ( TIM(6) )	20*20250
C	20*20255
C FILL IN UCON, TCON --- PRINT RESULTS	20*20260
C	20*20265
DO 60 I = 1, NMAX	20*20270
IF (LDC(I).EQ.1) GO TO 60	20*20275
TCON(I) = FMU * A(I)	20*20280

UCON(I) = (1./FMU) * BVAL(I)	20*20285
GO TO 50	20*20290
60 TCON(I) = BVAL(I)	20*20295
UCON(I) = A(I)	20*20300
50 CONTINUE	20*20305
WRITE (6,2000) TITL	20*20310
IF (ISTR5.EQ.0) WRITE (6,2050)	20*20315
IF (ISTR5.EQ.1) WRITE (6,2060)	20*20320
WRITE (6,2100)	20*20325
DO 70 I = 1,NSEG	20*20330
J = I + NSEG	20*20335
DO 40 N = 1,2	20*20340
80 PXY7(N) = (XYZ(I,1,N) + XYZ(I,2,N))/2.	20*20345
70 WRITE (6,2200) I, UCON(I), UCON(J), TCON(I), TCON(J),	20*20350
1 LDC(I), LDC(J), PXYZ(1), PXYZ(2)	20*20355
IF (IPUNCH.EQ.0) RETURN	20*20360
DO 120 I = 1,NSEG	20*20365
J = I + NSEG	20*20370
120 WRITE (7,2500) I, UCON(I), UCON(J)	20*20375
DO 130 I = 1,NSEG	20*20380
J = I + NSEG	20*20385
130 WRITE (7,2500) I, TCON(I), TCON(J)	20*20390
RETURN	20*20395
2000 FORMAT (1H1, 15A5 // 10X 19HBOUNDARY CONDITIONS)	20*20400
2050 FORMAT ( / 4( 18H PLANE STRAIN **** ) )	20*20405
2060 FORMAT ( / 4( 18H PLANE STRESS **** ) )	20*20410
2100 FORMAT (// 4X 4H SEG 7X 2HU1 10X 2HU2 10X 2HT1 10X 2HT2 8X 4HLDC1	20*20415
1 6X 4HLDC2 8X 2HX1 10X 2HX2 //)	20*20420
2200 FORMAT (2X I5, 2F12.8, 2F12.0, 6X I1, 11X I1, 2F12.6)	20*20425
2300 FORMAT (1H1, 16A5 // 10X 22HRIGHT HAND SIDE VECTOR //)	20*20430
2400 FORMAT (5X, I5, 2E15.8)	20*20435
2500 FORMAT ( 110, 2E30.10)	20*20440
3000 FORMAT (/// ( 2(8F12.6 /) /// )	20*20445
END	20*20450

SUBROUTINE DELINT (G, RHS)	20*25000
COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), TCON(200), LOC(200)	20*25005
COMMON / ARRAY2 / BVAL(200)	20*25010
COMMON / MATCON / FMU, POISN, P1, P1, P2, P3, P4, P5	20*25015
COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NHDYP	20*25020
COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, ISDY	20*25025
DIMENSION A(2), E1(2), E2(2), P(2), X(2,2), R1(2), R2(2)	20*25030
DIMENSION ISYM(2), U(2,2), T(2,2), G(NSIZE,NSIZE)	20*25035
DOUBLE PRECISION RHS(NSIZE), XI1, XI2, XI3, XI4, XI5	20*25040
DO 10 I = 1,NSIZE	20*25045
DO 10 J = 1,NSIZE	20*25050
10 G(I,J) = 0.	20*25055
DO 20 M = 1,NTOTAL	20*25060
IFLG = 0	20*25065
JFLAG = 0	20*25070
M1 = (M-1)/NSEG	20*25075
M2 = M - M1*NSEG	20*25080
M3 = M2 + NSEG	20*25085
C IF (LOC(M2).EQ.1.AND.ABS(BVAL(M2)).LT.1.0.AND.	20*25090
C 1 LOC(M3).EQ.1.AND.ABS(BVAL(M3)).LT.1.0) IFLG = 1	20*25095
C IF (LOC(M2).EQ.2.AND.ABS(BVAL(M2)).LT.1.0E-08.AND.	20*25100
C 1 LOC(M3).EQ.2.AND.ABS(BVAL(M3)).LT.1.0E-08) JFLAG = 1	20*25105
C	20*25110
C COMPUTE SYMMETRY COEFFICIENTS USING Y, THEN X	20*25115
C	20*25120
IFLAG = 1	20*25125
DO 16 K = 1,2	20*25130
J = 3 - K	20*25135
I = (M-1)/(NSEG*((2**J)/2))	20*25140
ISYM(K) = (-1)**I	20*25145
IF (I.EQ.0) ISYM(K) = 1	20*25150
16 IFLAG = IFLAG + ISYM(K)	20*25155
DO 30 J = 1,2	20*25160
IF (IFLAG.GT.0) GO TO 25	20*25165
X(1,J) = XYZ(M2,2,J) * ISYM(J)	20*25170
X(2,J) = XYZ(M2,1,J) * ISYM(J)	20*25175
GO TO 35	20*25180
25 X(1,J) = XYZ(M2,1,J) * ISYM(J)	20*25185
X(2,J) = XYZ(M2,2,J) * ISYM(J)	20*25190
35 CONTINUE	20*25195
C	20*25200
C DEFINE DIRECTION OF THE LINE SEGMENT F2 = A(J) / AMAG	20*25205
C	20*25210
30 A(J) = X(2,J) - X(1,J)	20*25215
AMAG = SQRT (A(1)**2 + A(2)**2)	20*25220
DO 33 I = 1,2	20*25225
E2(I) = A(I)/AMAG	20*25230
J = 3 - I	20*25235
33 E1(I) = F2(I) * (-1)**(J+1)	20*25240
C	20*25245
C CALCULATE THE ANGLES T1 AND T2 AND THE DISTANCE D	20*25250
C	20*25255
DO 20 N = 1,NSEG	20*25260
DO 15 J = 1,2	20*25265
P(J) = (XYZ(N,1,J) + XYZ(N,2,J))/2.	20*25270
R1(J) = X(1,J) - P(J)	20*25275
R2(I) = X(2,J) - P(J)	20*25280

DO 15 I = 1,2	20*25285
U(I,J) = 0.	20*25290
15 T(I,J) = 0.	20*25295
CALC DOTPRD (R1, E1, U)	20*25300
CALC DOTPRD (R1, E2, R12)	20*25305
CALL DOTPRD (R2, E2, R22)	20*25310
CALL DOTPRD (R1, R1, R1MAG)	20*25315
CALC DOTPRD (R2, R2, R2MAG)	20*25320
R1MAG = SQRT (R1MAG)	20*25325
R2MAG = SQRT (R2MAG)	20*25330
RA = ABS(R12)	20*25335
RB = ABS(R22)	20*25340
RMAC = AMAX1 (RA, RB)	20*25345
IF (ABS(D / RMAG).LT.1.0E-03) GO TO 40	20*25350
SIGN = D / ABS(D)	20*25355
T1 = ATAN(R12/D) - (1.-SIGN)*PI/2.	20*25360
T2 = ATAN(R22/D) - (1.-SIGN)*PI/2.	20*25365
ST1 = R12 / R1MAG	20*25370
ST2 = R22 / R2MAG	20*25375
CT1 = D / R1MAG	20*25380
CT2 = D / R2MAG	20*25385
TN1 = R12 / D	20*25390
TN2 = R22 / D	20*25395
C	20*25400
C DIAGNOSTIC PRINT --- OCCURS ONLY IN THE CASE OF SERIOUS DATA ERROR	20*25405
C	20*25410
IF ( (CT1/CT2) .GT. 0.) GOTO 500	20*25415
WRITE (6,2000) M, N, X, P, R1, R2, F1, E2, T1, T2, CT1, CT2, D	20*25420
500 CONTINUE	20*25425
XL1 = ALOG(D/CT1)	20*25430
XL2 = ALOG(D/CT2)	20*25435
C	20*25440
C CALCULATE DELU INTEGRAL FOR D.NE.0	20*25445
C	20*25450
IF (IFLG.EQ.1) GO TO 45	20*25455
X11 = D*(TN2*XL2-TN2+T2-TN1*XL1+TN1-T1)	20*25460
X12 = D*(T2-T1)	20*25465
X13 = D*(XL2-XL1)	20*25470
X14 = D*(TN2-TN1-T2+T1)	20*25475
DO 50 IX = 1,2	20*25480
DO 50 JX = IX,2	20*25485
DEL = 0.	20*25490
IF (IX.EQ.JX) DEL = 1.	20*25495
UXY = P1*(P2*DEL*X11-E1(IX)*E1(JX)*X12-(E1(IX)*E2(JX)+E1(JX)*E2(IX	20*25500
1 ))*X13-E2(IX)*E2(JX)*X14)	20*25505
U(IX,JX) = UXY * ISYM(JX)	20*25510
IF (IX.EQ.JX) GO TO 50	20*25515
U(JX,IX) = UXY * ISYM(IX)	20*25520
50 CONTINUE	20*25525
C	20*25530
C CALCULATE DELT INTEGRAL FOR D.NE.0	20*25535
C	20*25540
45 IF (JFLG.EQ.1) GO TO 75	20*25545
X11 = T2-T1	20*25550
X12 = T2+ST2*CT2-T1-ST1*CT1	20*25555
X13 = ST2**2-ST1**2	20*25560
X14 = T2-ST2*CT2-T1+ST1*CT1	20*25565
X15 = ALOG(CT1/CT2)	20*25570

DO 60 IX = 1,2	20*25575
DO 60 JX = IX,2	20*25580
TXY = 0.	20*25585
IF (IX.FQ.JX.AND.M.FQ.N) GO TO 60	20*25590
DEL = 0.	20*25595
IF (IX.FQ.JX) DEL = 1.	20*25600
TXY = P3*(P4*DEL*XI1+E1(IX)*E1(JX)*XI2+(E1(IX)*E2(JX)+E2(IX)*	20*25605
1 E1(JX))*XI3+E2(IX)*E2(JX)*XI4)	20*25610
T(IX,JX) = TXY * ISYM(JX)	20*25615
IF (IX.FQ.JX) GO TO 60	20*25620
TSTAR = -P3*P4*(E2(IX)*E1(JX)-E1(IX)*E2(JX))*XI5	20*25625
T(IX,JX) = (TXY+TSTAR)*ISYM(JX)	20*25630
T(JX,IX) = (TXY-TSTAR)*ISYM(IX)	20*25635
60 CONTINUE	20*25640
GO TO 75	20*25645
40 CONTINUE	20*25650
XI1 = R22*(ALOG(RB)-1.)-R12*(ALOG(RA)-1.)	20*25655
XI2 = R22 - R12	20*25660
XI3 = ALOG(RB) - ALOG(RA)	20*25665
C	20*25670
C CALCULATE DELU FOR D.EQ.U	20*25675
IF (IFLG.EQ.1) GO TO 65	20*25680
DO 70 IX = 1,2	20*25685
DO 70 JX = IX,2	20*25690
DEL = 0.	20*25695
IF (IX.FQ.JX) DEL = 1.	20*25700
UXY = P1*(P2*DEL*XI1-E2(IX)*E2(JX)*XI2)	20*25705
U(IX,JX) = UXY * ISYM(JX)	20*25710
IF (IX.FQ.JX) GO TO 70	20*25715
U(JX,IX) = UXY * ISYM(IX)	20*25720
70 CONTINUE	20*25725
C	20*25730
C CALCULATE DELT INTEGRAL FOR D.EQ.0	20*25735
65 IF (JFLG.EQ.1) GO TO 75	20*25740
DO 80 IX = 1,2	20*25745
DO 80 JX = IX,2	20*25750
IF (IX.FQ.JX) GO TO 80	20*25755
TXY = -P3*P4*(E2(IX)*E1(JX)-E1(IX)*E2(JX))*XI3	20*25760
T(IX,JX) = TXY * ISYM(JX)	20*25765
T(JX,IX) = -TXY * ISYM(IX)	20*25770
80 CONTINUE	20*25775
75 DO 85 IX = 1,2	20*25780
DO 85 JX = 1,2	20*25785
N4 = N + (IX-1)*NSFG	20*25790
M4 = M2 + (JX-1)*NSFG	20*25795
IF (IX.FQ.JX.AND.M.EQ.N) T(IX,JX) = -0.50	20*25800
IF (LOC(M4).EQ.1) GO TO 90	20*25805
TRANS = U(IX,JX)	20*25810
U(IX,JX) = -(1./FMU) * T(IX,JX)	20*25815
T(IX,JX) = -FMU * TRANS	20*25820
90 RHS(N4) = RHS(N4) + U(IX,JX) * BVAL(M4)	20*25825
85 G(N4,M4) = G(N4,M4) + T(IX,JX)	20*25830
20 CONTINUE	20*25835
RETURN	20*25840
2000 FORMAT (// 5X 215 / (2F10.5 /))	20*25845
END	20*25850

SUBROUTINE INSOLU( C )	20*30000
COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), TCON(200), LDC(200)	20*30005
COMMON / ARRAY3 / PTIN(100,2)	20*30010
COMMON / MATCON / FMU, POISN, P1, P1, P2, P3, P4, P5	20*30015
COMMON / TIMERS / TIM (10)	20*30020
COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NBDYP	20*30025
COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, IBDY	20*30030
DIMENSION C(100,3), A(4), PXYZ(3)	20*30035
IF (IBDY.NE.0) GO TO 100	20*30040
110 IF (NPT.EQ.0) RETURN	20*30045
CALL TIME ( TIM(7) )	20*30050
WRITE (6,2000) TITL	20*30055
IF (ISTRS.EQ.0) WRITE (6,2050)	20*30060
IF (ISTRS.EQ.1) WRITE (6,2060)	20*30065
C	20*30070
C CALL FOR CALCULATION OF DELD AND DELS	20*30075
C	20*30080
WRITE (6,2100)	20*30085
A(4) = 0.	20*30090
CALL DELSD (C)	20*30095
DO 10 NP = 1,NPT	20*30100
DO 20 I = 1,3	20*30105
20 A(I) = C(NP,I)	20*30110
IF (ISTRS.EQ.1) GO TO 30	20*30115
A(4) = POISN * (A(1) + A(3))	20*30120
30 CONTINUE	20*30125
THETA = (A(1) + A(3) + A(4))/3.	20*30130
TAUOCT = SQRT(2.*(A(1)**2+A(3)**2+A(4)**2-A(1)*A(3)-A(3)*A(4)-	20*30135
1 A(1)*A(4)+3.*A(2)**2))/3.	20*30140
WRITE (6,2200) NP,(A(K),K=1,4),THETA,TAUOCT,PTIN(NP,1),PTIN(NP,2)	20*30145
10 CONTINUE	20*30150
CALL TIME ( TIM(8) )	20*30155
RETURN	20*30160
100 WRITE (6,2000) TITL	20*30165
DO 120 I = 1,NSEG	20*30170
J = I + NSEG	20*30175
120 READ (5,1100) N, UCON(I), UCON(J)	20*30180
DO 130 I = 1,NSEG	20*30185
J = I + NSEG	20*30190
130 READ (5,1100) N, TCON(I), TCON(J)	20*30195
WRITE (6,2300)	20*30200
DO 140 I = 1,NSEG	20*30205
J = I + NSEG	20*30210
DO 150 N = 1,2	20*30215
150 PXYZ(N) = (XYZ(I,1,N) + XYZ(I,2,N))/2.	20*30220
140 WRITE (6,2400) I, UCON(I), UCON(J), TCON(I), TCON(J),	20*30225
1 LDC(I), LDC(J), PXYZ(1), PXYZ(2)	20*30230
GO TO 110	20*30235
1100 FORMAT (I10, 2E30.10)	20*30240
2000 FORMAT (1H1, 10X, 16A5)	20*30245
2050 FORMAT ( / 4/ 16H PLANE STRAIN **** ) )	20*30250
2060 FORMAT ( / 4/ 16H PLANE STRESS **** ) )	20*30255
2100 FORMAT (6HOPPOINT, 2X 10H SIGMA(XX) 2X 10H SIGMA(XY) 2X	20*30260
1 10H SIGMA(YY) 2X 10H SIGMA(ZZ) 4X 6H THETA 6X 7H TAUOCT	20*30265
2 5X 2H X 6X 2H Y)	20*30270
2200 FORMAT (2X 13, 2X 6F12.2, 2FA.4)	20*30275
2300 FORMAT (// 4X 4H SEG 7X 2HUI 10X 2HUI2 10X 2HT1 10X 2HT2 8X 4HLDC)	20*30280



1 6X 4HI,DC2 8X 2HX1 10X 2HX2 //)  
2400 FORMAT (2X I5, 2F12.8, 2F12.0, 6X I1, 11X I1, 2F12.6)  
END

20\*30285  
20\*30290  
20\*30295

SUBROUTINE DELSD (G)	20*35000
COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), TCON(200), LDC(200)	20*35005
COMMON / ARRAY3 / PTIN(100,2)	20*35010
COMMON / MATCON / FMU, PR, PI, C1, C2, C3, C4	20*35015
COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NRDYP	20*35020
COMMON / CONTR2 / TITL(16), IPUNCH, ISTR5, IADY	20*35025
DIMENSION A(2), E1(2), E2(2), P(2), X(2,2), R1(2), R2(2)	20*35030
DIMENSION ISYM(2), G(100,3)	20*35035
DO 10 I = 1,100	20*35040
DO 10 J = 1,3	20*35045
10 G(I,J) = 0.	20*35050
DO 20 M = 1,NTOTAL	20*35055
M1 = (M-1)/NSEG	20*35060
M2 = M - M1*NSEG	20*35065
C	20*35070
C COMPUTE SYMMETRY COEFFICIENTS USING Y, THEN X	20*35075
IFLAG = 1	20*35080
DO 16 K = 1,2	20*35085
J = 3 - K	20*35090
I = (M-1)/(NSEG*((2**J)/2))	20*35095
ISYM(K) = (-1)**I	20*35100
IF (I.F0.0) ISYM(K) = 1	20*35105
16 IFLAG = IFLAG * ISYM(K)	20*35110
DO 32 J = 1,2	20*35115
IF (IFLAG.GT.0) GO TO 23	20*35120
X(1,J) = XYZ(M2,2,J) * ISYM(J)	20*35125
X(2,J) = XYZ(M2,1,J) * ISYM(J)	20*35130
GO TO 35	20*35135
23 X(1,J) = XYZ(M2,1,J) * ISYM(J)	20*35140
X(2,J) = XYZ(M2,2,J) * ISYM(J)	20*35145
35 CONTINUE	20*35150
C	20*35155
C DEFINE DIRECTION OF THE LINE SEGMENT E2 = A(J)/AMAG	20*35160
C	20*35165
32 A(J) = X(2,J) - X(1,J)	20*35170
AMAG = SQRT (A(1)**2 + A(2)**2)	20*35175
DO 33 I = 1,2	20*35180
E2(I) = A(I)/AMAG	20*35185
J = 3 - I	20*35190
33 E1(J) = E2(I) * (-1)**(J+1)	20*35195
C	20*35200
C CALCULATE THE ANGLES T1 AND T2 AND THE DISTANCE D	20*35205
C	20*35210
DO 20 N = 1,NPT	20*35215
DO 15 J = 1,2	20*35220
P(J) = PTIN(N,J)	20*35225
R2(J) = X(2,J) - P(J)	20*35230
R1(J) = X(1,J) - P(J)	20*35235
15 CONTINUE	20*35240
D1 = 0.	20*35245
D2 = 0.	20*35250
DO 17 J=1,2	20*35255
D1 = D1 + R1(J)*R1(J)	20*35260
17 D2 = D2 + R2(J)*R2(J)	20*35265
D1 = SQRT(D1)	20*35270
D2 = SQRT(D2)	20*35275
CALC DOTPRD (R1, E1, U)	20*35280

CALL DOTPRD (R1, E2, R12)	20*35285
CALL DOTPRD (R2, E2, R22)	20*35290
CALL DOTPRD (R1, R1, R1MAG)	20*35295
CALL DOTPRD (R2, R2, R2MAG)	20*35300
R1MAG = SQRT (R1MAG)	20*35305
R2MAG = SQRT (R2MAG)	20*35310
RA = ABS(R12)	20*35315
RB = ABS(R22)	20*35320
RMAG = AMAX1(RA, RB)	20*35325
IF (ABS(D / RMAG).LT.1.0E-03) GO TO 40	20*35330
SIGN = D / ABS(D)	20*35335
T1 = ATAN(R12/D) - (1.-SIGN)*PI/2.	20*35340
T2 = ATAN(R22/D) - (1.-SIGN)*PI/2.	20*35345
S1 = R12 / R1MAG	20*35350
S2 = R22 / R2MAG	20*35355
C1 = D / R1MAG	20*35360
C2 = D / R2MAG	20*35365
XL1 = ALOG(D1)	20*35370
XL2 = ALOG(D2)	20*35375
40 L = 0	20*35380
DO 25 I = 1,2	20*35385
DO 25 J = 1,2	20*35390
L = L + 1	20*35395
DO 25 K = 1,2	20*35400
DELTK = 0.	20*35405
DELKJ = 0.	20*35410
DELIJ = 0.	20*35415
IF (I.EQ.K) DELIK = 1.	20*35420
IF (K.EQ.J) DELKJ = 1.	20*35425
IF (I.EQ.J) DELIJ = 1.	20*35430
IF (ABS(D/RMAG).LT.1.0E-03) GO TO 30	20*35435
DD1 = T2 - T1	20*35440
DD2 = XL2 - XL1	20*35445
DD3 = T2-T1+S2*C2-S1*C1	20*35450
DD4 = S2**2-S1**2	20*35455
DD5 = T2-T1-S2*C2+S1*C1	20*35460
DD6 = 2.*DD2-S2**2+S1**2	20*35465
US1 = DD3/D	20*35470
DS2 = DD4/D	20*35475
DS3 = DD5/D	20*35480
DS4 = DD6/D	20*35485
DS5 = 3.*DS1 + 2.*(S2*C2**3-S1*C1**3)/D	20*35490
DS6 = 2.*(C1**4-C2**4)/D	20*35495
DS7 = 4.*DS1 - DS5	20*35500
DS8 = 2.*(S2**4-S1**4)/D	20*35505
A1JK = DELIK*E1(J)+DELKJ*E1(I)-DELIJ*E1(K)	20*35510
B1JK = DELIK*E2(J)+DELKJ*E2(I)-DELIJ*E2(K)	20*35515
C1JK = F1(I)*E1(J)*F1(K)	20*35520
F1JK = F2(I)*E2(J)*E2(K)	20*35525
D1JK = F1(I)*E2(J)*F1(K)+E2(I)*E1(J)*E1(K)+E1(I)*E1(J)*E2(K)	20*35530
E1JK = F1(I)*E2(J)*F2(K)+E2(I)*E1(J)*E2(K)+E2(I)*E2(J)*E1(K)	20*35535
G1JK = C4*DFLIJ*E1(K)+E1(I)*E1(J)*E1(K)+PR*(DELIK*E1(J)+DELKJ*	20*35540
1 E1(I))	20*35545
H1JK = C4*DFLIJ*E2(K)+E1(I)*E2(J)*E1(K)+E2(I)*E1(J)*E1(K)+	20*35550
1 PR*(DELIK*E2(J)+DELKJ*E2(I))+2.*E1(I)*E1(J)*E2(K)-	20*35555
2 F1(I)*E2(J)*F1(K)-E2(I)*E1(J)*E1(K))	20*35560
O1JK = C4*E2(I)*E2(J)*E1(K)+PR*(F1(I)*F2(J)*E2(K)+E2(I)*F1(J)*	20*35565
1 F2(K))	20*35570

PIJK = C4*(DELIK*E1(J)+DELKJ*E1(I))-DELIJ*E1(K)*(1.-4.*PR)	20*35575
DU = C3*(C4*(AIJK*DD1+BIJK*DD2)+CIJK*DD3+DIJK*DD4+EIJK*DD5	20*35580
1 +FIJK*DD6)*ISYM(K)	20*35585
DS = 2.*FMU*C3*(6IJK*DS1+HIJK*DS2+OIJK*DS3+PIJK*DS4-CIJK*DS5	20*35590
1 -DIJK*DS6-EIJK*DS7-FIJK*DS8)*ISYM(K)	20*35595
GO TO 24	20*35600
30 CONTINUE	20*35605
BIJK = DELIK*E2(J)+DELKJ*E2(I)-DELIJ*E2(K)	20*35610
OIJK = C4*(2.*E1(K)*E2(I)+E1(J)*DELIK+E1(I)*DELKJ)+	20*35615
1 2.*PR*(E1(I)*E2(J)+E2(K)+E2(I)*E1(J)*E2(K))-DELIJ*E1(K)	20*35620
S1 = R12/RA	20*35625
S2 = R22/RB	20*35630
C	20*35635
C FOLLOWING IDIOT CARDS REQUIRED FOR 110A FORTRAN	20*35640
C	20*35645
ARG1 = RB	20*35650
ARG2 = RA	20*35655
DD2 = S2*ALOG(ARG1) - S1*ALOG(ARG2)	20*35660
DD6 = DD2	20*35665
DS9 = 1./RA-1./RB	20*35670
DD = C3*(C4*BIJK*DD2+2.*E2(I)*E2(J)*E2(K)*DD6)*ISYM(K)	20*35675
DS = 2.*FMU*C3*OIJK*DS9*ISYM(K)	20*35680
24 CONTINUE	20*35685
M4 = M2 + (K-1) * NSEG	20*35690
G(N,L) = G(N,L) + DD*TCN(M4) - DS*ICON(M4)	20*35695
25 CONTINUE	20*35700
20 CONTINUE	20*35705
RETURN	20*35710
END	20*35715

SUBROUTINE RDYSTR (C)	20*40000
COMMON / ARRAY1 / XYZ(160,2,2), UCON(200), TCON(200), LOC(200)	20*40005
COMMON / ARRAY4 / NBDY(50,3)	20*40010
COMMON / MATCON / FMU, POISN, P1, P1, P2, P3, P4, P5	20*40015
COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NBDYP	20*40020
COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, IBDY	20*40025
COMMON / TIMERS / TIM(10)	20*40030
DIMENSION A(2), E1(2), E2(2), P(3,2), R(2), DU(2), T(2), C(50,4)	20*40035
IF (NBDYP.EQ.0) RETURN	20*40040
CALL TIME ( TIM(9) )	20*40045
C1 = 1.-2.*POISN	20*40050
C2 = 1.-POISN	20*40055
WRITE (6,2000) TITL	20*40060
WRITE (6,2100) ((NBDY(I,J),J=1,3),I=1,NBDYP)	20*40065
WRITE (6,2000) TITL	20*40070
IF (ISTRS.EQ.0) WRITE (6,2050)	20*40075
IF (ISTRS.EQ.1) WRITE (6,2060)	20*40080
WRITE (6,2200)	20*40085
C	20*40090
C I0 = BASE SEGMENT NUMBER	20*40095
C I1 = REAR DIFFERENCE SEGMENT NUMBER	20*40100
C I2 = FORWARD DIFFERENCE SEGMENT NUMBER	20*40105
C	20*40110
DO 15 N = 1,NBDYP	20*40115
I0 = NBDY (N,1)	20*40120
I1 = NBDY (N,2)	20*40125
I2 = NBDY (N,3)	20*40130
DO 20 M = 1,2	20*40135
P(1,M) = (XYZ(I0,1,M) + XYZ(I0,2,M))/2.	20*40140
P(2,M) = (XYZ(I1,1,M) + XYZ(I1,2,M))/2.	20*40145
P(3,M) = (XYZ(I2,1,M) + XYZ(I2,2,M))/2.	20*40150
R(M) = P(3,M) - P(2,M)	20*40155
20 A(M) = XYZ(I0,2,M) - XYZ(I0,1,M)	20*40160
SMAG = SQRT(R(1)**2 + R(2)**2)	20*40165
AMAG = SQRT(A(1)**2 + A(2)**2)	20*40170
DO 25 M = 1,2	20*40175
E2(M) = A(M)/ AMAG	20*40180
K = 3 - M	20*40185
E1(K) = F2(M) * (-1)**(K+1)	20*40190
I3 = I1 + (M-1)*NSEG	20*40195
I4 = I2 + (M-1)*NSEG	20*40200
I5 = I0 + (M-1)*NSEG	20*40205
DU(M) = (UCON(I4) - UCON(I3))/SMAG	20*40210
25 T(M) = TCON(I5)	20*40215
M = 0	20*40220
DO 30 I = 1,2	20*40225
DO 30 J = 1,2	20*40230
M = M + 1	20*40235
DIJ = 0.	20*40240
IF (I.EQ.J) DIJ = 1.	20*40245
C(N,M) = (C1/(2.*C2))*(T(I)*E1(J) + T(J)*E1(I)) - (FMU/C2)*POISN*	20*40250
1 (E2(J)*DU(I) + E2(I)*DU(J))	20*40255
DO 30 K = 1,2	20*40260
A1 = E1(I)*F1(J)*E1(K) + E1(I)*E2(J)*E2(K) +	20*40265
1 E2(I)*F1(J)*E2(K) + E2(I)*E2(J)*E1(K)	20*40270
A2 = E1(I)*F1(J)*E2(K) - E1(I)*E2(J)*E1(K) - E2(I)*E1(J)*E1(K)	20*40275
A3 = E1(I)*E2(J)*E1(K) + E2(I)*E1(J)*E1(K)	20*40280

A4 = E2(I)*F2(J)*E2(K)	20*40285
C(N,M) = C(N,M) - (C1/(2.*C2))*DIJ*F1(K)*T(K) + (1./(2.*C2))*(A1*	20*40290
1 T(K)) + (Fmu/C2)*(C1*A2 + C2*A3 + 3.*A4)*Dij(K) -	20*40295
2 (Fmu/C2)*C1*DIJ*E2(K)*DU(K)	20*40300
30 CONTINUE	20*40305
IF (ISTR.EQ.1) GO TO 35	20*40310
C(N,4) = POISN * (C(N,1) + C(N,3))	20*40315
35 THETA = (C(N,1) + C(N,3) + C(N,4))/3.	20*40320
TAUOCT = SQRT(2.*(C(N,1)**2+C(N,3)**2+C(N,4)**2-C(N,1)*C(N,3)-	20*40325
1 C(N,3)*C(N,4)-C(N,1)*C(N,4)+3.*C(N,2)**2))/3.	20*40330
15 WRITE (6,2300) IO,(C(N,M),M=1,4),THETA,TAUOCT,P(1,1),P(1,2)	20*40335
CALC TIME ( TIM(10) )	20*40340
RETURN	20*40345
1000 FORMAT (24I3)	20*40350
2000 FORMAT (1H1, 10X, 16A5)	20*40355
2050 FORMAT ( / 4( 18H PLANE STRAIN **** ) )	20*40360
2060 FORMAT ( / 4( 18H PLANE STRESS **** ; )	20*40365
2100 FORMAT ( / 5X 11HBASE NUMBER 2X 11HREFAR NUMBER 3X 10HFWO NUMBER //	20*40370
1 ( 3I12 / ) )	20*40375
2200 FORMAT (7H0SEGMENT 2X 10H SIGMA(XX) 2X 10H SIGMA(XY) 2X	20*40380
1 10H SIGMA(YX) 2X 10H SIGMA(ZZ) 4X 6H THETA 6X 7H TAUOCT	20*40385
2 5X 2H X 6X 2H Y)	20*40390
2300 FORMAT (2X I3, 2X 6F12.2, 2F8.4)	20*40395
END	20*40400

SUBROUTINE DOTPRD (A, B, C)  
DIMENSION A(2), B(2)  
C = A(1)\*B(1) + A(2)\*B(2)  
RETURN  
END

20\*45000  
20\*45005  
20\*45010  
20\*45015  
20\*45020

SUBROUTINE SOLVER (N, X, F, A)	20*50000
DIMENSION A(N,N), X(N), F(N), XX(160)	20*50005
DOUBLE PRECISION X	20*50010
DO 10 I = 1, N	20*50015
F(I) = 0.0	20*50020
10 CONTINUE	20*50025
N1 = N - 1	20*50030
DO 40 I = 2, N	20*50035
DO 45 J = I, N	20*50040
IF (ABS(A(I-1,I-1)) .GT. 0.) 60 TO 45	20*50045
I1 = I - 1	20*50050
WRITE (6,510) I1	20*50055
RETURN	20*50060
45 CONTINUE	20*50065
CX = A(J,I-1) / A(I-1,I-1)	20*50070
K2 = I	20*50075
DO 50 K = I, N	20*50080
A(J,K2) = A(J,K2) - CX * A(I-1,K2)	20*50085
K2 = K2 + 1	20*50090
50 CONTINUE	20*50095
A(J,I-1) = CX	20*50100
55 CONTINUE	20*50105
60 CONTINUE	20*50110
C FORWARD PASS - OPERATE ON RIGHT HAND SIDE AS	20*50115
C ON MATRIX	20*50120
62 CONTINUE	20*50125
DO 70 I = 2, N	20*50130
DO 65 J = I, N	20*50135
X(J) = X(J) - X(I-1) * A(J,I-1)	20*50140
65 CONTINUE	20*50145
70 CONTINUE	20*50150
C	20*50155
C BACKWARD PASS - SOLVE FOR AX = B	20*50160
XX(N) = X(N) / A(N,N)	20*50165
DO 80 I = 1, N1	20*50170
SUM = 0.0	20*50175
I2 = N - I + 1	20*50180
DO 75 J = I2, N	20*50185
SUM = SUM + A(I2-1,J) * XX(J)	20*50190
75 CONTINUE	20*50195
XX(I2-1) = (X(I2-1)-SUM) / A(I2-1,I2-1)	20*50200
80 CONTINUE	20*50205
DO 90 I = 1, N	20*50210
F(I) = F(I) + XX(I)	20*50215
90 CONTINUE	20*50220
RETURN	20*50225
510 FORMAT(//1X 25#ERROR RETURN FROM SEQSOV I10.	20*50230
1 35#DIAGONAL TERM REDUCED TO ZERO / )	20*50235
END	20*50240



## 5.2 TWO DIMENSIONAL ANISOTROPIC BOUNDARY-INTEGRAL EQUATION METHOD

### 5.2.1 Formulation of the Field Equations

The present note concerns the application of the Boundary-Integral Method to the solution of two dimensional, plane stress problems for fully anisotropic, elastic materials. The nature of the equations is such that engineering notation for all field variables is convenient. The notation and theoretical development of the field equations follows from Lekhnitskii [1]<sup>1</sup>. The development of the boundary-integral equations follows the usual method outlined by Cruse [2]. The solution of the problem of unit loads in the x- and y-directions, called the fundamental solution will be first be obtained. Next, the Betti reciprocal work theorem will be used to obtain Somigliana's identities for internal displacements and stresses. Finally, the Boundary-Integral Equation will be obtained from the Somigliana displacement identity.

In the plane stress equations presented in this note, the non-zero stress components are  $\{\sigma_x, \sigma_y, \tau_{xy}\}$  and the corresponding strain components are  $\{\epsilon_x, \epsilon_y, \gamma_{xy}\}$ . The equilibrium equations for the stresses are:

$$\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0\end{aligned}\tag{1}$$

The strain components are subject to the single compatibility equation

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<sup>1</sup>Brackets refer to references at the end of this note.

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad (2)$$

which guarantees the existence of single-valued displacements,  $u_x$ ,  $u_y$  which are related to the strains by

$$\frac{\partial u_x}{\partial x} = \epsilon_x, \quad \frac{\partial u_y}{\partial y} = \epsilon_y \quad (3)$$

$$\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \gamma_{xy}$$

The constitutive law for the fully-anisotropic elastic material in plane stress can be given in matrix form as

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{26} \\ \beta_{16} & \beta_{26} & \beta_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (4)$$

The  $\beta_{ij}$ 's are the material compliances and are known to be the components of a fourth-order tensor, as the strains<sup>2</sup> and stresses are components of second order tensors. The tensor character of the compliances is basic for the application of the current results to composite materials, as discussed by Ashton et. al. [3].

The compliances may be given in terms of engineering material constants

$$\begin{aligned} \beta_{11} &= 1/E_x, & \beta_{12} &= -\nu_{xy}/E_x \\ \beta_{22} &= 1/E_y, & \beta_{16} &= \eta_{xy,x}/E_x \\ \beta_{26} &= \eta_{xy,y}/E_y, & \beta_{66} &= 1/G_{xy} \end{aligned} \quad (5)$$

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<sup>2</sup>Using  $\gamma_{xy}/2$  as the tensorial shear component.

For orthotropic materials  $\beta_{16} = \alpha_{26} = 0$ . For later reference the stiffness coefficients are now introduced but not put in engineering terms

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{16} \\ \alpha_{12} & \alpha_{22} & \alpha_{26} \\ \alpha_{16} & \alpha_{26} & \alpha_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (6)$$

The Airy Stress Function is now introduced such that its existence guarantees satisfaction of equilibrium, Eq (1)

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \quad (7)$$

Substitution of Eq (7) into Eq (4) and Eq (2) results in the following governing differential equation for  $F(x,y)$

$$\begin{aligned} \beta_{11} \frac{\partial^4 F}{\partial y^4} - 2\beta_{16} \frac{\partial^4 F}{\partial x \partial y^3} + (2\beta_{12} + \beta_{66}) \frac{\partial^4 F}{\partial x^2 \partial y^2} \\ - 2\beta_{26} \frac{\partial^4 F}{\partial x^3 \partial y} + \beta_{22} \frac{\partial^4 F}{\partial x^4} = 0 \end{aligned} \quad (8)$$

Characteristic surfaces along which  $F(x,y)$  can be integrated may be found by introducing the notation

$$z = x + \mu y; \quad \mu = a + ib, \quad i = \sqrt{-1} \quad (9)$$

Substitution of Eq (9) into Eq (8) reduces Eq (8) to

$$\frac{d^4 F}{dz^4} [\beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22}] = 0 \quad (10)$$

If we are to obtain non-trivial results to Eq (10)  $d^4F / dz^4 \neq 0$  which requires

$$\beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22} = 0 \quad (11)$$

Eq (11) is the characteristic equation for the material; Lekhnitskii shows that the four roots of Eq (11) are never real and are distinct so long as the material is not isotropic. We denote the roots  $\mu_j = a_j + ib_j$  ( $j=1,2$ ) and  $\bar{\mu}_j = a_j - ib_j$ . Lekhnitskii also shows that  $b_j > 0$ , from thermodynamic considerations. Thus the characteristic directions become

$$z_k = x + \mu_k y, \quad k = 1,2 \quad (12)$$

and their conjugates.

The general form of the stress function can be given by the relation

$$F(x,y) = 2R \{ F_1(z_1) + F_2(z_2) \} \quad (13)$$

Introducing the notation  $dF_k/dz_k$  (no summation on  $k$ ) =  $\phi_k(z_k)$  the stresses become

$$\begin{aligned} \sigma_x &= 2R \{ \mu_1^2 \phi_1'(z_1) + \mu_2^2 \phi_2'(z_2) \} \\ \sigma_y &= 2R \{ \phi_1'(z_1) + \phi_2'(z_2) \} \\ \tau_{xy} &= -2R \{ \mu_1 \phi_1'(z_1) + \mu_2 \phi_2'(z_2) \} \end{aligned} \quad (14)$$

where the prime denotes ordinary differentiation. The strains may be obtained from Eq (14) and integrated to obtain the displacements

$$\begin{aligned} u_x &= 2R \{ p_1 \phi_1(z_1) + p_2 \phi_2(z_2) \} \\ u_y &= 2R \{ q_1 \phi_1(z_1) + q_2 \phi_2(z_2) \} \end{aligned} \quad (15)$$

where

$$\begin{aligned} p_k &= \beta_{11} \mu_k^2 + \beta_{12} - \beta_{16} \mu_k \\ q_k &= \beta_{12} \mu_k + \beta_{22} / \mu_k - \beta_{26} \end{aligned} \quad (16)$$

Equations (14) and (15) together with traction boundary conditions

$$\begin{aligned} t_x &= \sigma_x n_x + \tau_{xy} n_y = g_1 \\ t_y &= \tau_{xy} n_x + \sigma_y n_y = g_2 \end{aligned} \quad (17)$$

or displacement boundary conditions

$$u_x = h_1 ; u_y = h_2 \quad (18)$$

constitute the mathematical problem to be solved.

### 5.2.2 Fundamental Solution: Point Force Problem

The basic relation for the development of integral equations for the solution of the anisotropic problem is the solution for a point force in the infinite anisotropic plane. Two such solutions will be required: A unit force in the x-direction, and a unit force in the y-direction. Utilizing the traction formulae (17) it is easily shown that on an arbitrary closed surface

$$\begin{aligned} \oint_S t_x dS &= 2R [\mu_1 \phi_1 + \mu_2 \phi_2] \\ \oint_S t_y dS &= -2R [\phi_1 + \phi_2] \end{aligned} \quad (19)$$

where  $[[ \ ]]$  denotes the jump in the enclosed quantities for a full cycle of  $S$ . If the path  $S$  encloses the point of load application,  $z_0 = x_0 + i y_0$ , then the results of (19) will be non-zero.

Let  $\phi_{jk}$  represent the stress function for a point load in the  $x_j^3$  direction. The path integrals in (19) are seen to be of the opposite sign to the applied loads;

$$\begin{aligned} 2R [\phi_{j1} + \phi_{j2}] &= \delta_{j2} \\ 2R [\mu_1 \phi_{j1} + \mu_2 \phi_{j2}] &= -\delta_{j1} \end{aligned} \quad (20)$$

<sup>3</sup>We will now use indicial notation  $(x,y) = (x_1, x_2)$  and its associated conventions. The index  $k$  will never be summed.

for the point load solutions. Functions which satisfy (20) for any closed path around  $z_0$  are

$$\phi_{jk} = A_{jk} \log (z_k - z_{k_0}) \quad (21)$$

where  $z_{k_0} = x_0 + \mu_k y_0$ . In what follows  $z_{k_0}$  will, for convenience only, be taken as the origin,  $z_{k_0} = 0$ . It may be shown by suitable investigation near  $z_k = 0$  that (21) satisfies the requirements of a point force [4]. Since it is easily shown that

$$[\log z_k] = 2\pi i, \quad i = \sqrt{-1} \quad (22)$$

(20) leads to the result

$$A_{j1} - \bar{A}_{j1} + A_{j2} - \bar{A}_{j2} = \delta_{j2}/2\pi i \quad (23)$$

$$\mu_1 A_{j1} - \bar{\mu}_1 \bar{A}_{j1} + \mu_2 A_{j2} - \bar{\mu}_2 \bar{A}_{j2} = -\delta_{j1}/2\pi i$$

It is also required that the displacement field surrounding the applied be single-valued. That is

$$[u_j] = 0 \quad j = 1, 2 \quad (24)$$

Substitution of (21) into (15) and taking the jump around a closed path we find in addition to (23)

$$p_1 A_{j1} - \bar{p}_1 \bar{A}_{j1} + p_2 A_{j2} - \bar{p}_2 \bar{A}_{j2} = 0 \quad (25)$$

$$q_1 A_{j1} - \bar{q}_1 \bar{A}_{j1} + q_2 A_{j2} - \bar{q}_2 \bar{A}_{j2} = 0$$

Together, (23) and (25) are sufficient to find  $A_{jk}$ . Taking the notation

$$\mu_k = \alpha_k + i\gamma_k \quad i = \sqrt{-1} \quad (26)$$

$$A_{jk} = C_{jk} + i D_{jk}$$

it is easily shown that (23) and (25) together with (16) reduce in real form to

$$\begin{bmatrix} (2\beta_{11}\alpha_1\gamma_1 - \beta_{13}\gamma_1) & [\beta_{11}(\alpha_1^2 - \gamma_1^2) + \beta_{12} - \beta_{13}\alpha_1] \\ [\beta_{12}\gamma_1 - \beta_{22}\gamma_1/(\alpha_1^2 + \gamma_1^2)] & [\beta_{12}\alpha_1 + \beta_{22}\alpha_1/(\alpha_1^2 + \gamma_1^2) - \beta_{23}] \\ 0 & 1 \\ \gamma_1 & \alpha_1 \end{bmatrix} \quad (27)$$

$$\begin{bmatrix} (2\beta_{11}\alpha_2\gamma_2 - \beta_{13}\gamma_2) & [\beta_{11}(\alpha_2^2 - \gamma_2^2) + \beta_{12} - \beta_{13}\alpha_2] \\ [\beta_{12}\gamma_2 - \beta_{22}\gamma_2/(\alpha_2^2 + \gamma_2^2)] & [\beta_{12}\alpha_2 + \beta_{22}\alpha_2/(\alpha_2^2 + \gamma_2^2) - \beta_{23}] \\ 0 & 1 \\ \gamma_2 & \alpha_2 \end{bmatrix} \begin{Bmatrix} C_{j1} \\ D_{j1} \\ C_{j2} \\ D_{j2} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ -\delta_{j2}/4\pi \\ \delta_{j1}/4\pi \end{Bmatrix}$$



Equation (27) becomes singular when  $\mu_1 = \mu_2 = i$ , but it will be shown that  $A_{jk}$  may be found for very nearly isotropic materials.

We now define two tensor fields: The first is  $U_{jk}$  and corresponds to the displacements for the stress function (21) according to (15)

$$U_{ji} = 2R \{P_{i1}A_{j1} \log z_1 + P_{i2}A_{j2} \log z_2\} \quad (28)$$

where  $P_{1k} = p_k$ ,  $P_{2k} = q_k$ . Taking the derivatives of (21) at  $z_k$  according to (14) and substituting into (17), tractions on an arbitrary surface are found

$$T_{ji} = 2R \{Q_{i1}(\mu_1 n_1 - n_2) A_{j1}/z_1 + Q_{i2}(\mu_2 n_1 - n_2) A_{j2}/z_2\} \quad (29)$$

where

$$[Q_{ik}] = \begin{bmatrix} \mu_1 & \mu_2 \\ -1 & -1 \end{bmatrix} \quad (30)$$

### 5.2.3 Boundary-Integral Equation

Since the governing partial differential equation (10) admits no real characteristic surface the problem is elliptic and the stresses and displacements are continuous. Under such circumstances it is easily verified that Betti's reciprocal work theorem at the surface must be valid

$$\int_{S+\Gamma} T_{ji} u_i dS = \int_{S+\Gamma} U_{ji} t_i dS \quad (31)$$

The surface  $\Gamma$  is a circle of vanishing radius  $\epsilon$  surrounding the point load; it is added to exclude the singularity from the volume. The second integral in (31) is convergent as  $\epsilon \rightarrow 0$ . For continuous  $u_i$  it is sufficient to investigate the behavior of the integral

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma} T_{ji} dS \quad (32)$$

At a circle centered at the applied load it is seen that

$$\mu_k n_1 - n_2 = -(\mu_k \cos \theta - \sin \theta) \quad (33)$$

and  $dS = \epsilon d\theta$ ,  $0 \leq \theta < 2\pi$ . Extracting from (32) the variable part it is sufficient to find

$$\int_0^{2\pi} \frac{\mu_k n_1 - n_2}{z_k} dS = - \int_0^{2\pi} \frac{\mu_k \cos \theta - \sin \theta}{\mu_k \sin \theta + \cos \theta} d\theta \quad (34)$$

which, upon rearrangement becomes

$$\int_0^{2\pi} \frac{\mu_k n_1 - n_2}{z_k} dS = \mu_k \int \frac{d(\tan \theta)}{1 - \mu_k^2 \tan^2 \theta} - (1 + \mu_k^2) \int \frac{\cos \theta d(\cos \theta)}{(1 + \mu_k^2) \cos^2 \theta - \mu_k^2} \quad (35)$$

Equation (35) can then be integrated directly to obtain

$$\int_0^{2\pi} \frac{\mu_k n_1 - n_2}{z_k} dS = - \log [\cos \theta - \mu_k \sin \theta] \Big|_0^{2\pi} \quad (36)$$

Taking the real part of the argument of the log

$$\rho_k^2 = (\cos \theta - a_k \sin \theta)^2 + (b_k \sin \theta)^2 \quad (37)$$

and the imaginary part

$$\theta_k = - \tan^{-1} \left[ \frac{b_k \sin \theta}{\cos \theta - a_k \sin \theta} \right] \quad (38)$$

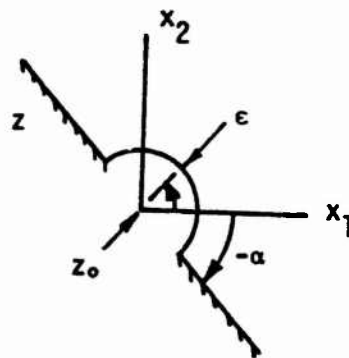
the result to (36) is found to be

$$\int_0^{2\pi} \frac{\mu_k n_1 - n_2}{z_k} dS = 2\pi i \quad (39)$$

Substitution of (39) into (32) and using (23) leads to the usual Somigliana identity for the interior displacement

$$\begin{aligned}
 u_j(z_0) = & - \int_S T_{ji}(z_k, z_0) u_i(z_k) dS(z_k) \\
 & + \int_S U_{ji}(z_k, z_0) t_i(z_k) dS(z_k)
 \end{aligned}
 \tag{40}$$

The Boundary-Integral Equation is found in the usual way [2] by allowing  $z_0$  to approach the boundary from the inside and evaluating the jumps in the singular integrals in (40). A simple means for evaluating the jump is to place  $z_0$  at the surface, augment the surface as shown in the figure and integrate (40).



In this case  $\mu_k n_1 - n_2 = \mu_k \cos \theta + \sin \theta$  since the normal  $(n_1, n_2)$  points outward from  $z_0$ . The range on  $\theta$  is  $-\alpha \leq \theta < \pi - \alpha$ . Again the only significant integral is in the first integral in (40)

$$\int_{-\alpha}^{\pi-\alpha} \frac{\mu_k n_1 - n_2}{z_k} \epsilon d\theta = \int_{-\alpha}^{\pi-\alpha} \frac{\mu_k \cos \theta - \sin \theta}{\mu_k \sin \theta - \cos \theta} d\theta
 \tag{41}$$

which, by the same steps as above, becomes

$$\int_{-\alpha}^{\pi-\alpha} \frac{\mu_k n_1 - n_2}{z_k} \epsilon d\theta = -i \left[ \tan^{-1} \left( \frac{b_k \sin \theta}{\cos \theta - a_k \sin \theta} \right) \right]_{-\alpha}^{\pi-\alpha} \quad (42)$$

Substitution of the limits in (42) yields

$$\int_{-\alpha}^{\pi-\alpha} \frac{\mu_k n_1 - n_2}{z_k} \epsilon d\theta = -i\pi \quad (43)$$

Again using the relations (23), (40) becomes

$$u_j/2 + \int_S T_{ji} u_i dS = \int_S U_{ji} t_i dS \quad (44)$$

Equation (44) is the Boundary-Integral Equation which relates unknown boundary data to known boundary data. Once (44) has been solved numerically (5V), (40) can be used to obtain interior displacements and stresses.

#### 5.2.4 Somigliana's Identity for Interior Strains, Stresses

The displacement gradient tensor  $u_{j,\ell}$  can be calculated from (40) by differentiation at  $z_0$ . Since  $\partial/\partial x_\ell [\log(z_k - z_0)] = -\partial/\partial x_{\ell_0}$   $[\log(z_k - z_0)]$  the differentiation may be written in terms of derivatives at  $z_k$  by a change in sign. Then

$$\frac{\partial u_j}{\partial x_\ell} = \int_S \frac{\partial T_{ji}}{\partial x_\ell} u_i dS - \int_S \frac{\partial U_{ji}}{\partial x_\ell} t_i dS \quad (45)$$

The tensorial strain at  $z_0$  is given by the symmetric part of (45)

$$\epsilon_{j\ell} = \frac{1}{2} (u_{j,\ell} + u_{\ell,j}) \quad (46)$$

such that

$$2\epsilon_{j\ell} = \int_S \left[ \frac{\partial T_{ji}}{\partial x_\ell} + \frac{\partial T_{\ell i}}{\partial x_j} \right] u_i dS - \int_S \left[ \frac{\partial U_{ji}}{\partial x_\ell} + \frac{\partial U_{\ell i}}{\partial x_j} \right] t_i dS \quad (47)$$

The kernels,  $S_{j\ell i}$ ,  $D_{j\ell i}$  respectively are given by

$$\begin{aligned} S_{j\ell i} = & -2R \{ R_{\ell 1} Q_{i1} (\mu_1 n_1 - n_2) A_{j1}/z_1^2 + R_{\ell 2} Q_{i2} (\mu_2 n_1 - n_2) A_{j2}/z_2^2 \} \\ & -2R \{ R_{j1} Q_{i1} (\mu_1 n_1 - n_2) A_{\ell 1}/z_1^2 + R_{j2} Q_{i2} (\mu_2 n_1 - n_2) A_{\ell 2}/z_2^2 \} \\ D_{j\ell i} = & 2R \{ R_{\ell 1} P_{i1} A_{j1} / z_1 + R_{\ell 2} P_{i2} A_{j2} / z_2 \} \\ & +2R \{ R_{j1} P_{i1} A_{\ell 1} / z_1 + R_{j2} P_{i2} A_{\ell 2} / z_2 \} \end{aligned} \quad (48)$$

where  $R_{1k} = 1$ ,  $R_{2k} = u_k$ . It is assumed that the boundary is piecewise flat. Then (47) becomes

$$2\epsilon_{j\ell} = \int_S S_{j\ell i} u_i dS - \int_S D_{j\ell i} t_i dS \quad (49)$$

The stresses can be determined by substitution of (49), with

$$\gamma_{xy} = 2\epsilon_{12}, \quad \epsilon_x = \epsilon_{11}, \quad \epsilon_y = \epsilon_{22}, \quad \text{into (6).}$$

### 5.2.5 Numerical Solution

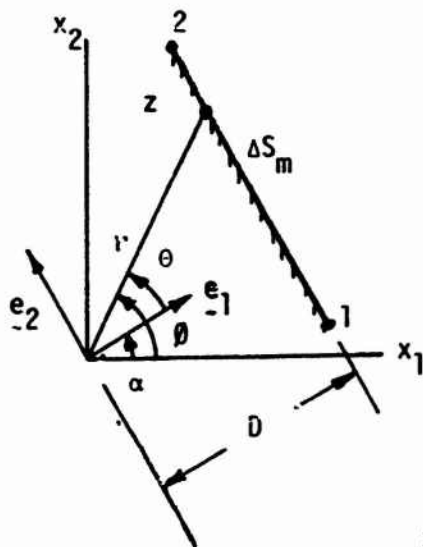
Following the procedure used in the isotropic theory, the boundary displacements and boundary tractions are assumed to be piecewise constant. When this is assumed over the  $M$  boundary segments (again taken as straight line segments), (44) becomes

$$u_j(n)/2 + \sum_{m=1}^M u_i(m) \int_{\Delta S_m} T_{ji}(n,m) dS = \sum_{m=1}^M t_i(m) \int_{\Delta S_m} U_{ji}(n,m) dS \quad (50)$$

Similarly the internal strains (49) become

$$\begin{aligned} 2\epsilon_{j\ell}(z_0) = & \sum_{m=1}^M u_i(m) \int_{\Delta S_m} S_{j\ell i}(z_0,m) dS \\ & - \sum_{m=1}^M t_i(m) \int_{\Delta S_m} D_{j\ell i}(z_0,m) dS \end{aligned} \quad (51)$$

By specifying the orientation of each line segment,  $\Delta S_m$ , with its normal  $(n_1, n_2)$  the integrals in (50) and (51) may be solved for explicitly. The notation is defined in the figure below



$$\begin{aligned} z_k &= r(\cos\theta + \mu_k \sin\theta) \\ D &= \underline{r} \cdot \underline{e}_1 \\ dS &= Dd(\tan\theta) \end{aligned} \quad (52)$$



The coordinate system ( $\underline{e}_1, \underline{e}_2$ ) is taken such that  $\underline{e}_1$  is in the direction of the *outward* normal and  $\underline{e}_2$  is tangent to  $\Delta S_m$ , going from "1" to "2" keeping the material on the left. Since  $\vartheta = \theta + \alpha$ , where  $\alpha = \cos^{-1}(n_1)$ , (52) can be written

$$z_k = r \cos\theta (\cos\alpha - \sin\alpha \tan\theta + \mu_k \cos\alpha \tan\theta + \mu_k \sin\alpha) \quad (53)$$

so that

$$z_k = D [(\cos\alpha + \mu_k \sin\alpha) + (\mu_k \cos\alpha - \sin\alpha) \tan\theta] \quad (54)$$

The closed-form integrals for (50), (51) are now easily obtained, as the only variable is  $\tan\theta$ . Two special cases,  $D = 0$  and the multivaluedness of  $\log z_k$  will be discussed below. Defining integrals of the variables in the kernels with  $\Delta I$ 's we obtain

$$\Delta U_{ji} = \int_{\Delta S} U_{ji} dS = 2R \{P_{i1} A_{j1} \Delta I_{11} + P_{i2} A_{j2} \Delta I_{12}\} \quad (55)$$

$$\begin{aligned} \Delta T_{ji} = \int_{\Delta S} T_{ji} dS = 2R \{ & Q_{i1} (\mu_1 n_1 - n_2) A_{j1} \Delta I_{21} \\ & + Q_{i2} (\mu_1 n_1 - n_2) A_{j2} \Delta I_{22} \} \end{aligned}$$

and for the strains

$$\begin{aligned}\Delta D_{j\ell i} = & 2R\{R_{\ell 1}P_{i1}A_{j1}\Delta I_{31} + R_{\ell 2}P_{i2}A_{j2}\Delta I_{32}\} \\ & + 2R\{R_{j1}P_{i1}A_{\ell 1}\Delta I_{31} + R_{j2}P_{i2}A_{\ell 2}\Delta I_{32}\}\end{aligned}\quad (56)$$

$$\begin{aligned}\Delta S_{j\ell i} = & -2R\{R_{\ell 1}Q_{i1}(\mu_1 n_1 - n_2)A_{j1}\Delta I_{41} + R_{\ell 2}Q_{i2}(\mu_2 n_1 - n_2)A_{j2}\Delta I_{42}\} \\ & - 2R\{R_{j1}Q_{i1}(\mu_1 n_1 - n_2)A_{\ell 1}\Delta I_{41} + R_{j2}Q_{i2}(\mu_2 n_1 - n_2)A_{\ell 2}\Delta I_{42}\}\end{aligned}$$

The integrals are easily calculated for  $D \neq 0$

$$\begin{aligned}\Delta I_1 &= \int_1^2 \log z_k dS = \frac{1}{\mu_k \cos \alpha - \sin \alpha} [z_k (\log z_k - 1)] \Big|_1^2 \\ \Delta I_2 &= \Delta I_3 \int_1^2 \frac{dS}{z_k} = \frac{1}{\mu_k \cos \alpha - \sin \alpha} (\log z_k) \Big|_1^2 \\ \Delta I_4 &= \int_1^2 \frac{dS}{z_k^2} = \frac{1}{\mu_k \cos \alpha - \sin \alpha} \left( -\frac{1}{z_k} \right) \Big|_1^2\end{aligned}$$

The case for  $D = 0$  may be deduced as a special result of (57) or by integrating again, with proper substitutions for  $z_k$  and  $dS$ . Using " $\pm$ " to denote the cases where  $dS = +dr$  and  $dS = -dr$  we note  $\theta = \pm\pi/2$  and

$$z_k = \pm r (\mu_k \cos \alpha - \sin \alpha) \quad (58)$$

Substitution of (58) into the integrals in (55), (56) we obtain

$$\begin{aligned}\Delta I_1 &= \int_1^2 \log[\pm r(u_k \cos \alpha - s \sin \alpha)] (\pm dr) = \frac{1}{u_k \cos \alpha - s \sin \alpha} [z_k (\log z_k - 1)] \Big|_1^2 \\ \Delta I_2 = \Delta I_3 &= \int_1^2 \frac{dS}{z_k} = \frac{1}{u_k \cos \alpha - s \sin \alpha} (\log z_k) \Big|_1^2 \\ \Delta I_4 &= \int_1^2 \frac{dS}{z_k^2} = -\frac{1}{(u_k \cos \alpha - s \sin \alpha)^2} \left( \pm \frac{1}{r} \right) \Big|_1^2\end{aligned}\quad (59)$$

The first two results in equation (57) contain the term  $\log z_k$ . The log is multivalued, and has a jump of  $+2\pi$  on the digital computer as  $\theta$  passes from  $\pi-\epsilon$  to  $\pi+\epsilon$ . To account for this on the computer the change in the imaginary part of  $\log z_k$  is tested. If a change of more than  $\pi$  in the imaginary part is found then the result is corrected by adding  $(\pm 2\pi)$  to the imaginary part of the  $\log z_k$ .

### 5.2.6 Usage Guide for ANISOT

The computer program is divided into three major sections, along the same lines as the solution just detailed. The first section solves the boundary-integral equations (44), producing a fully known set of boundary tractions and displacements. The next section utilizes Somigliana's identity (51) to obtain stresses at specified interior points. The final section determines stresses at the surface of the body, using the tractions obtained in section one, and the tangential derivatives of the displacements.

The boundary solution may be output on punched cards if desired. This option allows the user to input the boundary solution directly, and the program will begin execution of sections two and three.

#### 5.2.6.1 Problem Size Specifications

The program allows up to two degrees of symmetry of geometry and boundary conditions. Present array dimensions limit the number of boundary segments to 80 (320 with symmetry), but capacity may be increased by changing the common statements labeled ARRAY1 and ARRAY2. A number of other cards should also be modified, and are listed below, by card number:

ANI10065	ANI15235
ANI10070	ANI20035
ANI10100	ANI5005
ANI15050	

The number of interior points which may be specified is limited to 200, while the limit on surface stress points is 50. These limits may be increased by changing ARRAY3 and ARRAY4, respectively.

The time required for execution of each subroutine is calculated through an assembly language subroutine, TIME. Users of computers other than the Univac 1108 should provide their own routine for this purpose, or insert a dummy routine, TIME(T).

#### 5.2.6.2 *Specification of Material Constants*

Material constants are specified through four matrices, listed below:

- STIFF, stiffness matrix (6)
- FLEX, compliance matrix (4)
- MU, solutions to the characteristic equation (11)
- AX, coefficients of the net load terms in  
Lekhnitskii's stress function (see §II)

The coefficients AX are calculated in a program called AXCALC, which serves as an auxiliary program. All the matrices described above may be obtained as punched output from AXCALC, and inserted directly into the data deck.

### 5.2.6.3 Use of AXCALC

This program makes use of a capability previously detailed [5] to solve the characteristic equation for a generally anisotropic material, symmetric about its mid-plane. The algorithm is specifically designed for a layered material, since major use will be found in the area of advanced fiber composites. AXCALC requires as input only the stacking sequence of the laminate, and the material properties of the individual laminae, in their principal material directions.

Input data required by AXCALC is summarized in the table below. Items which appear on the same card are bracketed, and the format for each card appears opposite the first item on that card.

INPUT FOR AXCALC		
ITEM	DESCRIPTION	FORMAT
NC	Number of laminates to be analyzed	I3
TITL	Title card - any 80 characters	16A5
NANG	Number of individual laminae	I3
[ E11	Major Young's modulus (single lamina)	(3E15.10, F10.4)
E22	Minor Young's modulus	
G12	In-plane shear modulus	
V12	Principal Poisson's ratio	
THETA(I)	Orientation angles of the individual laminae (angles measured from the x axis to the major axis of the layer)	10F8.3
THICK(I)	Thicknesses of individual laminae	8F10.8

#### 5.2.6.4 Identification of Parameters in ANISOT

All parameters necessary for use of the integral equation program are defined below. For easy reference, the more important ones are defined on cards numbered ANI5065 - ANI5120 in subroutine SETUP. All the descriptions below are summarized in Table 2, and parameters which may be described concisely appear only in Table 2.

NODE (I,J) - a temporary array which stores the two node numbers associated with each segment number ('rear' number, then 'forward' number)

XYZM (I,J) - a temporary array containing the  $x_1$  and  $x_2$  coordinates of each node, in that order.

PTIN (I,J) - coordinates of the interior stress solution points ( $x_1, x_2$ ). These are read only if NPT  $\neq 0$ .

NBDY (I,J) - three segment numbers necessary for the surface stress solution, read only if NBDYP  $\neq 0$ . For each segment on which a stress solution is desired, three segment numbers are read, in the following order: segment number on which stress solution is desired; segment number for the "rear" difference value of  $\Delta u / \Delta s$ ; segment number for the "forward" difference value of  $\Delta u / \Delta s$ . The 'forward' direction

is taken as the positive "s" direction, always directed along the boundary of the body with the material on the left.

#### 5.2.6.5 Boundary Conditions

Both traction and displacement boundary conditions are possible, and are input by means of a NAMELIST read statement. The boundary data is preceded and followed by control cards, as shown below:

```
_$ BDYCON  
  { DATA  
_$ END
```

All boundary conditions are initialized to zero and the boundary condition key, LDC(I), is initialized to 1, meaning a traction boundary condition is assumed for each segment. Setting LDC = 2 means a displacement boundary condition will be specified for the given segment and direction. Traction conditions are specified by a parameter TCON(I), while displacement conditions are specified by UCON(I).

All  $x_1$ -direction boundary data is stored, followed by  $x_2$ -direction data. The value of the subscript I for the boundary parameters (UCON, TCON, LDC) is determined in the following manner.

For data specified in the  $x_1$  direction,

I = the segment number (N), but for the  $x_2$  direction

I = N + NSEG, where NSEG is the total number of boundary segments.

For example consider a body which has been represented by 24 segments.



Tractions specified on segment 6:

TCON(6) = .100E04, (x<sub>1</sub> traction)

TCON(30) = .325E04, (x<sub>2</sub> traction)

Displacements specified on segment 9:

LDC(9) = 2, LDC(33) = 2,

UCON(9) = 0.001, (x<sub>1</sub> displacement)

UCON(33) = 0.004, (x<sub>2</sub> displacement)

Displacements set at zero on segment 1:

LDC(1) = 2, (x<sub>1</sub>)

LDC(25) = 2, (x<sub>2</sub>)

### 5.2.6.6 Input Data for ANISOT

ITEM	DESCRIPTION	FORMAT
NC	Number of problems to be solved	I3
TITL	Title card -any 80 characters	16A5
NSEG	Number of segments on the boundary	10I5
NSYM	Degrees of symmetry (y, then x)	
NNOD	Number of boundary nodes connecting segments	
IPUNCH	= 0, the boundary solution will not be punched	
ISTR	= 0, plane strain ; = 1 plane stress	
IBDY	≠ 0, boundary data read from cards	
NPT	Number of interior points for stress solution	
NDDYP	Number of points for boundary stresses	
NODE(I,J)	Nodes associated with each segment number	24I3
XYZM(I,J)	( $x_1, x_2$ ) coordinates of each node	16F5.3
STIFF*	Stiffness matrix (6)	6E13.7
FLEX*	Compliance matrix (4)	6E13.7
MU*	Solutions of the characteristic equation (11)	4E20.10
AX(I,J)*	Coefficients of stress function (see §II)	4E20.10
{ Boundary	a NAMELIST read statement	
{ Conditions	(See standard references for format)	
PTIN(I,J)	Interior Stress Solution points ( $x_1, x_2$ )	16F5.3
NBDY(I,J)	Segment numbers associated with surface stress solutions.	24I3

\*Cards available as direct output of AXCALC

### 5.2.7 References

- [1] S. G. Lekhnitskii, *Theory of Elasticity of an Elastic Body*, Holden-Day (1963).
- [2] T. A. Cruse, *Application of the Boundary-Integral Equation Method to Problems of Solid Mechanics* (In preparation).
- [3] J. E. Ashton, J. C. Halpin, P. H. Petit, *Primer on Composite Materials: Analysis*, Technomic (1969).
- [4] E. Sternberg, R. A. Eubanks, "On the Concept of Concentrated Loads and an Extension of the Uniqueness Theorem in the Linear Theory of Elasticity", *Journal of Rational Mechanics and Analysis*, 4 (1955).
- [5] H. J. Konish, Jr., "Numerical Calculation of the Characteristic Parameters for a Generally Anisotropic Plate - MULTMU Usage Guide", Report SM-65, Carnegie-Mellon University, (June 1971).

## 5.2.8 Listing for AXCALC Computer Program

C	DETERMINES COEFFICIENTS OF NET LOAD TERMS FOR ANISOTROPIC STRESS FTM	AXC10000
	COMMON / MATCON / E11, E22, G12, V12, E(3,3), BETA(3,3)	AXC10005
	COMMON / GEOMTY / THETA(10), THICK(10), NANG, PI	AXC10010
	COMMON / ROOTS1 / LAMDA(20)	AXC10015
	DIMENSION STIF(6), FLEX(6), DELTA(2,2)	AXC10020
	DIMENSION TITL(16), C(4,4), R(4,4), RHS(4), X(4), R(4), XI(4,4)	AXC10025
	COMPLEX LAMDA	AXC10030
C		AXC10035
C	NC = NUMBER OF CASES TO BE SOLVED SEQUENTIALLY	AXC10040
C	N = ORDER OF CHARACTERISTIC EQUATION.	AXC10045
C	THETA = ANGLE FROM THE X-AXIS TO THE 1-AXIS, IN DEGREES.	AXC10050
C	E11, E22, V12, AND G12 ARE THE MATERIAL PROPERTIES OF THE	AXC10055
C	INDIVIDUAL LAMINAE.	AXC10060
C	K = NUMBER OF LAMINAE IN THE LAMINATE.	AXC10065
C	THICK = THICKNESS OF EACH LAMINA IN THE LAMINATE.	AXC10070
C		AXC10075
	PI = 3.1415926536	AXC10080
	DO 15 I = 1,2	AXC10085
	DO 15 J = 1,2	AXC10090
	DELTA(I,J) = 0.0	AXC10095
15	IF (I.EQ.J) DELTA(I,J) = 1.0	AXC10100
	READ(5,100) NC	AXC10105
45	WRITE(6,105) NC	AXC10110
	NC = NC - 1	AXC10115
	IF (NC.LT.0) STOP	AXC10120
	READ(5,110) TITL	AXC10125
	READ(5,120) NANG	AXC10130
	READ(5,125) E11,E22,G12,V12	AXC10135
	READ(5,130) (THETA(I), I = 1,NANG)	AXC10140
	READ(5,135) (THICK(I), I = 1,NANG)	AXC10145
	CALC MILTMU	AXC10150
	K = 0	AXC10155
	DO 25 I = 1,3	AXC10160
	DO 25 J = 1,3	AXC10165
	K = K + 1	AXC10170
	STIF(K) = E(I,J)	AXC10175
25	FLEX(K) = BETA(I,J)	AXC10180
	WRITE (7,1000) STIF	AXC10185
	WRITE (7,1000) FLEX	AXC10190
	WRITE(6,115) TITL	AXC10195
	WRITE(6,140)	AXC10200
	WRITE(6,145) E11, E22, G12, V12	AXC10205
	WRITE(6,150)	AXC10210
	WRITE(6,155) (THETA(L), L = 1,NANG)	AXC10215
	WRITE(6,160)	AXC10220
	WRITE(6,165) (THICK(L), L = 1,NANG)	AXC10225
	WRITE(6,180)	AXC10230
	WRITE(6,175) ((E(I,J),J = 1,3), I = 1,3)	AXC10235
	WRITE(6,190)	AXC10240
	WRITE(6,175) ((BETA(I,J), J = 1,3), I = 1,3)	AXC10245
	WRITE(6,195)	AXC10250
	WRITE(6,200) (LAMDA(I), I = 1,4)	AXC10255
	ALPH1 = REAL ( LAMDA(1) )	AXC10260
	GAM1 = ABS( AIMAG( LAMDA(1) ) )	AXC10265
	ALPH2 = REAL ( LAMDA(3) )	AXC10270
	GAM2 = ABS( AIMAG( LAMDA(3) ) )	AXC10275
	WRITE(7,280) ALPH1, GAM1, ALPH2, GAM2	AXC10280

C(1,1) = 2.*HETA(1,1)*ALPH1*GAM1 - HETA(1,3)*GAM1	AXC10285
C(1,3) = 2.*HETA(1,1)*ALPH2*GAM2 - HETA(1,3)*GAM2	AXC10290
C(1,2) = HETA(1,1)*(ALPH1**2-GAM1**2) + BETA(1,2) - HETA(1,3)*ALPH1	AXC10295
C(1,4) = HETA(1,1)*(ALPH2**2-GAM2**2) + BETA(1,2) - BETA(1,3)*ALPH2	AXC10300
C(2,1) = HETA(1,2)*GAM1 - HETA(2,2)*GAM1/(ALPH1**2+GAM1**2)	AXC10305
C(2,2) = HETA(1,2)*ALPH1 + BETA(2,2)*ALPH1/(ALPH1**2+GAM1**2)	AXC10310
1 - HETA(2,3)	AXC10315
C(2,3) = HETA(1,2)*GAM2 - BETA(2,2)*GAM2/(ALPH2**2+GAM2**2)	AXC10320
C(2,4) = HETA(1,2)*ALPH2 + BETA(2,2)*ALPH2/(ALPH2**2+GAM2**2)	AXC10325
1 - HETA(2,3)	AXC10330
C(3,1) = 0.0	AXC10335
C(3,3) = 0.0	AXC10340
C(3,2) = 1.0	AXC10345
C(3,4) = 1.0	AXC10350
C(4,1) = GAM1	AXC10355
C(4,3) = GAM2	AXC10360
C(4,2) = ALPH1	AXC10365
C(4,4) = ALPH2	AXC10370
DO 35 I = 1,2	AXC10375
DO 35 J = 1,4	AXC10380
35 C(I,J) = C(I,J) * E22	AXC10385
DO 05 I=1,4	AXC10390
DO 05 J=1,4	AXC10395
05 B(I,J) = C(I,J)	AXC10400
CALL INVR (C, 4, DUMMY, 0, DET, 4, 4)	AXC10405
DO 10 I = 1,4	AXC10410
DO 10 J = 1,4	AXC10415
XI(I,J) = 0.0	AXC10420
DO 10 K = 1,4	AXC10425
10 XI(I,J) = XI(I,J) + C(I,K) * B(K,J)	AXC10430
WRITE(6,250)	AXC10435
WRITE(6,210) ((XI(I,J),J=1,4),I=1,4)	AXC10440
DO 40 K = 1,2	AXC10445
RHS(1) = 0.0	AXC10450
RHS(2) = 0.0	AXC10455
RHS(3) = -DFLTA(K,2)/(4.*PI)	AXC10460
RHS(4) = DFLTA(K,1)/(4.*PI)	AXC10465
WRITE(6,260)	AXC10470
WRITE(6,210) (RHS(I), I = 1,4)	AXC10475
DO 20 I = 1,4	AXC10480
X(I) = 0.0	AXC10485
DO 20 J = 1,4	AXC10490
20 X(I) = X(I) + C(I,J) * RHS(J)	AXC10495
WRITE(6,270) (X(I),I=1,4)	AXC10500
WRITE(7,280) (X(I),I=1,4)	AXC10505
DO 30 I = 1,4	AXC10510
R(I) = 0.0	AXC10515
DO 30 J = 1,4	AXC10520
30 R(I) = R(I) + B(I,J) * X(J)	AXC10525
WRITE(6,260)	AXC10530
WRITE(6,210) (R(I),I=1,4)	AXC10535
40 CONTINUE	AXC10540
GO TO 45	AXC10545
100 FORMAT(I3)	AXC10550
105 FORMAT(1H1,I3)	AXC10555
110 FORMAT(16A5)	AXC10560
115 FORMAT(1H1,16A5)	AXC10565
120 FORMAT(2I3)	AXC10570

125	FORMAT(3F15.10,F10.4)	AXC10575
130	FORMAT(10F8.3)	AXC10580
135	FORMAT(4F10.8)	AXC10585
140	FORMAT(1H0.36X,'E11',7X,'E22',7X,'G12',7X,'V12',7X,'V21')	AXC10590
145	FORMAT(1H.30X,5E10.4)	AXC10595
150	FORMAT(1H0.30X,'ORIENTATION, TOP TO BOTTOM, IN DEGREES')	AXC10600
155	FORMAT(4AX,F10.5)	AXC10605
160	FORMAT(1H0.30X,'LAMINA THICKNESS, TOP TO BOTTOM, IN INCHES')	AXC10610
165	FORMAT(4AX,F10.7)	AXC10615
170	FORMAT(1H0.30X,'THE A-MATRIX FOR THIS LAMINATE IS')	AXC10620
175	FORMAT( (30X,3(E14.5,5X)),//)	AXC10625
180	FORMAT(1H0.30X,'THE E-MATRIX FOR THIS LAMINATE IS')	AXC10630
190	FORMAT(1H0.30X,'THE PLANE-STRESS BETA MATRIX IS')	AXC10635
195	FORMAT(1H0.20X,'THE MU-VALUES FOR THIS LAMINATE ARE')	AXC10640
200	FORMAT( 2(5X,F20.12),'J')	AXC10645
210	FORMAT((/4E14.8) )	AXC10650
250	FORMAT ( // // // // , 10X , ' THE IDENTITY MATRIX IS ' , // )	AXC10655
260	FORMAT( // // // // , 10X , ' THE RHS VECTOR ' , // // )	AXC10660
270	FORMAT ( // // // // , 10X , ' A1 = ',2E14.7,'J',/ 10X,'A2 = ',2E14.7,	AXC10665
	1 'J', // // )	AXC10670
280	FORMAT (4F20.10)	AXC10675
1000	FORMAT (6E13.7)	AXC10680
	END	AXC10685

SUBROUTINE MILTMU	AXC15000
COMMON / MATCON / E11, E22, G12, V12, E(3,3), BETA(3,3)	AXC15005
COMMON / GEOMTY / THETA(10), THICK(10), NANG, PI	AXC15010
COMMON / ROOTS1 / LAMDA(20)	AXC15015
DIMENSION QB(10,3,3), A(3,3), COFAC(3,3), TRANS(3,3)	AXC15020
DIMENSION D(11)	AXC15025
COMPLEX LAMDA	AXC15030
K = NANG	AXC15035
V21 = E22*V12/E11	AXC15040
Q11 = F11/(1.0 - V12*V21)	AXC15045
Q22 = F22/(1.0 - V12*V21)	AXC15050
Q12 = V21*Q11	AXC15055
Q66 = G12	AXC15060
H = 0.0	AXC15065
DO 10 L = 1,K	AXC15070
H = H + THICK(L)	AXC15075
10 CONTINUE	AXC15080
DO 15 L = 1,K	AXC15085
THRAD = PI*THETA(L)/180.0	AXC15090
T = COS(THRAD)	AXC15095
S = SIN(THRAD)	AXC15100
QB(1,1,1) = Q11*(T**4) + 2.*(Q12 + 2.*Q66)*((S*T)**2) + Q22*(S**4)	AXC15105
QB(1,2,2) = Q11*(S**4) + 2.*(Q12 + 2.*Q66)*((S*T)**2) + Q22*(T**4)	AXC15110
QB(1,1,2) = (Q11 + Q22 - 4.*Q66)*((S*T)**2) + Q12*((S**4) + (T**4))	AXC15115
QB(1,1,3) = (Q11 - Q12 - 2.*Q66)*S*(T**3) + (Q12 - Q22 + 2.*Q66)*T*	AXC15120
1 (S**3)	AXC15125
QB(1,2,3) = (Q11 - Q12 - 2.*Q66)*T*(S**3) + (Q12 - Q22 + 2.*Q66)*S*	AXC15130
1 (T**3)	AXC15135
QB(1,3,3) = (Q11 + Q22 - 2.*(Q12 + Q66))*((S*T)**2) + Q66*((S**4)	AXC15140
1 + (T**4))	AXC15145
QB(1,2,1) = QB(L,1,2)	AXC15150
QB(1,3,2) = QB(L,2,3)	AXC15155
QB(1,3,1) = QB(L,1,3)	AXC15160
15 CONTINUE	AXC15165
DO 20 I = 1,3	AXC15170
DO 25 J = 1,3	AXC15175
A(I,J) = 0.0	AXC15180
DO 30 L = 1,K	AXC15185
ASUM = QB(L,I,J)*THICK(L)	AXC15190
A(I,J) = A(I,J) + ASUM	AXC15195
30 CONTINUE	AXC15200
E(I,J) = A(I,J)/H	AXC15205
25 CONTINUE	AXC15210
20 CONTINUE	AXC15215
DET = (F(1,1)*E(2,2)*E(3,3)) + (F(1,2)*F(2,3)*E(3,1)) +	AXC15220
1 (F(1,3)*E(2,1)*E(3,2)) - (E(3,1)*E(2,2)*E(1,3)) -	AXC15225
2 (F(3,2)*E(2,3)*E(1,1)) - (E(3,3)*E(2,1)*E(1,2))	AXC15230
COFAC(1,1) = (E(2,2)*E(3,3)) - (E(3,2)*E(2,3))	AXC15235
COFAC(1,2) = -1.0*((E(2,1)*E(3,3)) - (E(3,1)*E(2,3)))	AXC15240
COFAC(1,3) = (E(2,1)*E(3,2)) - (E(3,1)*E(2,2))	AXC15245
COFAC(2,1) = -1.0*((E(1,2)*E(3,3)) - (E(3,2)*F(1,3)))	AXC15250
COFAC(2,2) = (F(1,1)*E(3,3)) - (E(3,1)*E(1,3))	AXC15255
COFAC(2,3) = -1.0*((E(1,1)*E(3,2)) - (F(3,1)*E(1,2)))	AXC15260
COFAC(3,1) = (E(1,2)*E(2,3)) - (E(2,2)*E(1,3))	AXC15265
COFAC(3,2) = -1.0*((E(1,1)*E(2,3)) - (E(2,1)*E(1,3)))	AXC15270
COFAC(3,3) = (E(1,1)*E(2,2)) - (F(2,1)*E(1,2))	AXC15275
DO 35 I = 1,3	AXC15280

```

DO 40 J = 1,3
TRANS(I,J) = COFAC(J,I)
HETA(I,J) = (TRANS(I,J))/DET
40 CONTINUE
35 CONTINUE
D(1) = HETA(1,1)
D(2) = -2.0*HETA(1,3)
D(3) = 2.0*HETA(1,2) + HETA(3,3)
D(4) = -2.0*HETA(2,3)
D(5) = HETA(2,2)
CALL ROOTS(D,4,LAMDA)*
RETURN
END

```

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AXC15285
AXC15290
AXC15295
AXC15300
AXC15305
AXC15310
AXC15315
AXC15320
AXC15325
AXC15330
AXC15335
AXC15340
AXC15345

```

\*Standard root solving routine called here. User must supply such a routine.



```

SUBROUTINE INVR (A, N, B, M, D, IS, JS)
C
C A = MATRIX, DIMENSIONS SHOWN, IN WHICH A SUBMATRIX IN UPPER LEFT-
C HAND CORNER IS TO BE INVERTED
C N = ORDER OF SUBMATRIX
C D = DETERMINANT
C M = 0, THEN
C B = B(1,1) IN CALLING ROUTINE, AND INVERSE STORED IN A
C M = 1, THEN
C B = B(N,1) IN CALLING ROUTINE, AND A-INVERSE * B RETURNED IN B
C M = M (.GT. 1), THEN
C B = B(N,M) IN CALLING ROUTINE, AND A-INVERSE * B RETURNED IN B
C A-INVERSE IS NOT DESTROYED IN LAST TWO CASES

```

```

C
C DIMENSION A(IS,JS), B(1,1), IN(100,2), IP(100), P(100)
C D = 1.0
C DO 03 J = 1,N
03 IP(J) = 0
C DO 06 I = 1,N
C AMAX = 0.0
C DO 18 J = 1,N
C IF (IP(J) - 1) 06, 18, 06
06 DO 15 K = 1,N
C IF (IP(K) - 1) 09, 15, 72
09 IF (ABS(AMAX) - ABS(A(J,K))) 12, 15, 15
12 IR = J
C IC = K
C AMAX = A(J,K)
15 CONTINUE
18 CONTINUE
C IP(IC) = IP(IC) + 1
C IF (IR - IC) 21, 33, 21
21 D = -D
C DO 24 L = 1,N
C SWAP = A(IR,L)
C A(IR,L) = A(IC,L)
24 A(IC,L) = SWAP
C IF (M) 33, 33, 27
27 DO 30 L = 1,M
C SWAP = B(IR,L)
C B(IR,L) = B(IC,L)
30 B(IC,L) = SWAP
33 IN(1,1) = IR
C IN(1,2) = IC
C P(I) = A(IC,IC)
C D = D * P(I)
C A(IC,IC) = 1.0
C DO 36 L = 1,N
36 A(IC,L) = A(IC,L) / P(I)
C IF (M) 45, 45, 39
39 DO 42 L = 1,M
42 B(IC,L) = B(IC,L) / P(I)
45 DO 40 L1 = 1,N
C IF (L1 - IC) 48, 60, 48
48 T = A(L1,IC)
C A(L1,IC) = 0.0
C DO 51 L = 1,N

```

```

AXC20000
AXC20005
AXC20010
AXC20015
AXC20020
AXC20025
AXC20030
AXC20035
AXC20040
AXC20045
AXC20050
AXC20055
AXC20060
AXC20065
AXC20070
AXC20075
AXC20080
AXC20085
AXC20090
AXC20095
AXC20100
AXC20105
AXC20110
AXC20115
AXC20120
AXC20125
AXC20130
AXC20135
AXC20140
AXC20145
AXC20150
AXC20155
AXC20160
AXC20165
AXC20170
AXC20175
AXC20180
AXC20185
AXC20190
AXC20195
AXC20200
AXC20205
AXC20210
AXC20215
AXC20220
AXC20225
AXC20230
AXC20235
AXC20240
AXC20245
AXC20250
AXC20255
AXC20260
AXC20265
AXC20270
AXC20275
AXC20280

```

51	A(L1,L) = A(L1,L) - A(IC,L) * T	AXC20285
	IF (M) 60, 60, 64	AXC20290
54	DO 57 L = 1,M	AXC20295
57	B(L1,L) = B(L1,L) - B(IC,L) * T	AXC20300
60	CONTINUE	AXC20305
	DO 69 T = 1,N	AXC20310
	L = N + 1 - I	AXC20315
	IF (IN(L,1) - IN(L,2)) 63, 69, 63	AXC20320
63	JR = IN(L,1)	AXC20325
	JC = IN(L,2)	AXC20330
	DO 66 K = 1,N	AXC20335
	IF (N.F0.0) JR=K	AXC20340
	SWAP = A(K,JR)	AXC20345
	A(K,JR) = A(K,JC)	AXC20350
66	A(K,JC) = SWAP	AXC20355
69	CONTINUE	AXC20360
72	RETURN	AXC20365
100	FORMAT (//10F13.5)	AXC20370
300	FORMAT (1H1)	AXC20375
	END	AXC20380

### 5.2.9 Listing for ANISOT Computer Program

C		ANTI0000
C	MAIN PROGRAM -- INITIALIZES DATA - CALLS SUBROUTINES	ANTI0005
C		ANTI0010
	COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), TCON(200), LOC(200)	ANTI0015
	COMMON / ARRAY2 / BVAL(200)	ANTI0020
	COMMON / MATCON / P1,FMU,PQ(2,2),MU(2),FLEX(6),STIF(6),AX(2,2)	ANTI0025
	COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSTZE, NPT, NBDYP	ANTI0030
	COMMON / CONTR2 / TITL(16), IPUNCH, ISTR5, IRDY	ANTI0035
	COMMON / TIMERS / T (10)	ANTI0040
	COMPLEX PQ, MU, AX	ANTI0045
C		ANTI0050
C	THE DIMENSIONS OF THE FOLLOWING ARRAYS ARE PROBLEM DEPENDENT	ANTI0055
C		ANTI0060
	DIMENSION C(160,160)	ANTI0065
	DOUBLE PRECISION RHS(160)	ANTI0070
	READ (5,100) NC	ANTI0075
	5 WRITE (6,200) NC	ANTI0080
	NC = NC - 1	ANTI0085
	IF (NC .LT. 0) STOP	ANTI0090
	PI = 3.14159265	ANTI0095
	DO 10 I = 1,200	ANTI0100
	UCON(I) = 0.	ANTI0105
	TCON(I) = 0.	ANTI0110
	10 BVAL(I) = 0.	ANTI0115
	CALC TIME ( T(1) )	ANTI0120
	DO 20 I = 2,10	ANTI0125
	20 T(I) = 0.	ANTI0130
	CALC SETUP	ANTI0135
	IF (IRDY.NE.0) GO TO 30	ANTI0140
	CALC HVSOLU (C, RHS)	ANTI0145
	30 CALC INSOLU (C)	ANTI0150
	CALC BDYSTR (C)	ANTI0155
C		ANTI0160
C	CALCULATE TIME CHART	ANTI0165
C		ANTI0170
	T(2) = (T(2)-T(1))*10**(-3)	ANTI0175
	T(4) = (T(4)-T(3))*10**(-3)	ANTI0180
	T(6) = (T(6)-T(5))*10**(-3)	ANTI0185
	T(8) = (T(8)-T(7))*10**(-3)	ANTI0190
	T(10) = (T(10)-T(9))*10**(-3)	ANTI0195
	WRITE (6,2000) TITL	ANTI0200
	WRITE (6,2100)	ANTI0205
	WRITE (6,2200) T(2), T(4), T(6), T(8), T(10)	ANTI0210
	GO TO 5	ANTI0215
	100 FORMAT ( 13 )	ANTI0220
	200 FORMAT ( 1H1, 5X I3)	ANTI0225
	1000 FORMAT ( 16A5)	ANTI0230
	2000 FORMAT (1H1, 16A5)	ANTI0235
	2100 FORMAT ( 21H TIME BREAKDOWN CHART //)	ANTI0240
	2200 FORMAT ( 5X 15HIME FOR SETUP F12.7, 2X 7HSECONDS //	ANTI0245
	1 5X 15HIME FOR DELINT F12.7, 2X 7HSECONDS //	ANTI0250
	2 5X 15HIME FOR SOLVER F12.7, 2X 7HSECONDS //	ANTI0255
	3 5X 15HIME FOR INSOLU F12.7, 2X 7HSECONDS //	ANTI0260
	4 5X 15HIME FOR BDYSOL F12.7, 2X 7HSECONDS)	ANTI0265
	END	ANTI0270

# SUBROUTINE SETUP

```

COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), ICON(200), LDC(200)
COMMON / ARRAY2 / BVAL(200)
COMMON / ARRAY3 / PTIN(100,2)
COMMON / ARRAY4 / NBDY(50,3)
COMMON / MATCON / PI,PMU,PQ(2,2),MU(2),FLEX(6),STIF(6),AX(2,2)
COMMON / TIMFRS / TIM(10)
COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NBDYP
COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, IBDY
NAMELIST / HBYCON / UCON, TCON, LDC
DIMENSION NODE(100,2), XYZM(100,2)
COMPLEX PQ, MU, AX
EQUVALFNCE (NODE, LDC), (XYZM, UCON)

```

```

C
C NSEG = NUMBER OF SEGMENTS ON THE BOUNDARY
C NSYM = NUMBER OF DEGREES OF SYMMETRY STARTING WITH Y, THEN Y
C NNOD = NUMBER OF BOUNDARY NODES CONNECTING BOUNDARY SEGMENTS
C IPUNCH = 1 -- THE BOUNDARY SOLUTION WILL BE PUNCHED OUT
C ISTRS = 0, PLSTRN -- ISTRS = 1, PLSTRS
C IF IBDY.EQ.0 --- BOUNDARY DATA STORED IN COMMON
C IF IBDY.NE.0 --- BOUNDARY DATA READ IN FROM CARDS ADDED TO END
C OF THE DATA DECK
C NPT = NUMBER OF INTERIOR SOLUTION POINTS FOR STRESS SOLUTION
C NBDYP = NUMBER OF BOUNDARY POINTS FOR STRESS SOLUTION
C
READ (5,1000) TITL
READ (5,1100) NSEG, NSYM, NNOD, IPUNCH, ISTRS, IBDY, NPT, NBDYP
READ (5,1200) ((NODE(I,J),J=1,2),I=1,NSFG)
READ (5,1300) ((XYZM(I,J),J=1,2),I=1,NNOD)
READ (5,1400) STIF
READ (5,1400) FLEX
READ (5,1700) MU
READ (5,1700) ((AX(I,J),J=1,2),I=1,2)
WRITE (6,2000) TITL
WRITE (6,2100) NSEG, NSYM, NNOD, IPUNCH, ISTRS, IBDY, NPT, NBDYP
WRITE (6,2200) ((NODE(I,J),J=1,2),I=1,NSFG)
WRITE (6,2300) ((XYZM(I,J),J=1,2),I=1,NNOD)
WRITE (6,2400) STIF
WRITE (6,2400) FLEX
WRITE (6,2700) MU
WRITE (6,2700) ((AX(I,J),J=1,2),I=1,2)
NSIZE = 2 * NSEG
DO 10 I = 1, NSEG
DO 10 J = 1, 2
DO 10 K = 1, 2
N = NODE(I,J)
10 XYZ(I,J,K) = XYZM(N,K)
DO 20 I = 1, 200
UCON(I) = 0.
20 LDC(I) = 1
READ (5,HBYCON)
IF (NPT.EQ.0) GO TO 30
READ (5,1500) ((PTIN(I,J),J=1,2),I=1,NPT)
WRITE (6,2500) ((PTIN(I,J),J=1,2),I=1,NPT)
30 IF (NBDYP.EQ.0) GO TO 40
READ (5,1600) ((NBDY(I,J),J=1,3),I=1,NBDYP)
WRITE (6,2600) ((NBDY(I,J),J=1,3),I=1,NBDYP)

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ANI15000  
 ANI15005  
 ANI15010  
 ANI15015  
 ANI15020  
 ANI15025  
 ANI15030  
 ANI15035  
 ANI15040  
 ANI15045  
 ANI15050  
 ANI15055  
 ANI15060  
 ANI15065  
 ANI15070  
 ANI15075  
 ANI15080  
 ANI15085  
 ANI15090  
 ANI15095  
 ANI15100  
 ANI15105  
 ANI15110  
 ANI15115  
 ANI15120  
 ANI15125  
 ANI15130  
 ANI15135  
 ANI15140  
 ANI15145  
 ANI15150  
 ANI15155  
 ANI15160  
 ANI15165  
 ANI15170  
 ANI15175  
 ANI15180  
 ANI15185  
 ANI15190  
 ANI15195  
 ANI15200  
 ANI15205  
 ANI15210  
 ANI15215  
 ANI15220  
 ANI15225  
 ANI15230  
 ANI15235  
 ANI15240  
 ANI15245  
 ANI15250  
 ANI15255  
 ANI15260  
 ANI15265  
 ANI15270  
 ANI15275  
 ANI15280

40 CONTINUE	ANT15285
NFAC = 2**NSYM	ANT15290
IF (NSYM.EQ.0) NFAC = 1	ANT15295
NTOTAL = NSEG * NFAC	ANT15300
C	ANT15305
C CALCULATE NEEDED MATERIAL CONSTANTS	ANT15310
C	ANT15315
DO 50 K = 1,2	ANT15320
PQ(1,K) = FLEX(1)*MU(K)**2 + FLEX(2) - FLEX(3)*MU(K)	ANT15325
50 PQ(2,K) = FLEX(2)*MU(K) + FLEX(4)/MU(K) - FLEX(5)	ANT15330
FMU = STIF(1)	ANT15335
CALI TIME ( TIM(2) )	ANT15340
RETURN	ANT15345
1000 FORMAT (16A5)	ANT15350
1100 FORMAT (10I5)	ANT15355
C	ANT15360
C ***** CAUTION***** FORMATS PROBLEM DEPENDENT ***** CAUTION *****	ANT15365
C	ANT15370
1200 FORMAT (24I3)	ANT15375
1300 FORMAT (16F5.3)	ANT15380
1400 FORMAT ( 6E13.7 )	ANT15385
1500 FORMAT (16F5.3)	ANT15390
1600 FORMAT (24I3)	ANT15395
1700 FORMAT(4E20.10)	ANT15400
2000 FORMAT (1H1, 10X, 16A5)	ANT15405
2100 FORMAT (// 10I5)	ANT15410
2200 FORMAT (// 8(3X 2I3))	ANT15415
2300 FORMAT (// 4(3X 2F10.6))	ANT15420
2400 FORMAT ( 10X 3E12.7 /, 22X 2E12.7 /, 34X E12.7 )	ANT15425
2500 FORMAT (// 4(3X 2F10.6))	ANT15430
2600 FORMAT (// 6(3X 3I3))	ANT15435
2700 FORMAT ( 2( 2F10.6, 8X ) )	ANT15440
2800 FORMAT ( 5X, 8E12.6 )	ANT15445
END	ANT15450

SUBROUTINE HVSOLU (C, RHS)	ANT20000
COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), TCON(200), LDC(200)	ANT20005
COMMON / ARRAY2 / BVAL(200)	ANT20010
COMMON / MATCON / PI, FMI, PQ(2,2), MU(2), FLEX(6), STIF(6), AV(2,2)	ANT20015
COMMON / CONTR1 / NSEG, NSYM, NIOFAL, NSIZE, NPT, NBDYP	ANT20020
COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, IDBY	ANT20025
COMMON / TIMERS / TIM (10)	ANT20030
DIMENSION A(200), PXYZ(2), C(NSIZE,NSIZE)	ANT20035
EQUVALENCE (A, UCON)	ANT20040
DOUBLE PRECISION RHS(NSIZE)	ANT20045
COMPLEX PQ, MU, AX	ANT20050
NMAX = 2 * NSEG	ANT20055
WRITE (6,2000) TITL	ANT20060
IF (ISTRS.EQ.0) WRITE (6,2050)	ANT20065
IF (ISTRS.EQ.1) WRITE (6,2060)	ANT20070
WRITE (6,2100)	ANT20075
C	ANT20080
C WRITE THE STARTING BOUNDARY CONDITIONS	ANT20085
C	ANT20090
DO 10 I = 1, NSEG	ANT20095
J = I + NSEG	ANT20100
DO 15 N = 1, 2	ANT20105
15 PXYZ(N) = (XYZ(1,1,N) + XYZ(1,2,N))/2.	ANT20110
10 WRITE (6,2200) I, UCON(I), UCON(J), TCON(I), TCON(J),	ANT20115
1 LDC(I), LDC(J), PXYZ(1), PXYZ(2)	ANT20120
DO 20 I = 1, NMAX	ANT20125
RHS(I) = 0.000	ANT20130
IF (LDC(I).EQ.1) GO TO 30	ANT20135
BVAL(I) = FMI * UCON(I)	ANT20140
GO TO 20	ANT20145
30 BVAL(I) = TCON(I)	ANT20150
20 CONTINUE	ANT20155
C	ANT20160
C CALCULATE DELU, DELT, RHS	ANT20165
C	ANT20170
CALI TIME ( TIM(3) )	ANT20175
CALI DFLINT (C, RHS)	ANT20180
CALI TIME ( TIM(4) )	ANT20185
C	ANT20190
C WRITE RIGHT HAND SIDE VECTOR	ANT20195
C	ANT20200
WRITE (6,2300) TITL	ANT20205
DO 40 I = 1, NSEG	ANT20210
J = I + NSEG	ANT20215
40 WRITE (6,2400) I, RHS(I), RHS(J)	ANT20220
C	ANT20225
C SOLVE SYSTEM OF EQUATIONS	ANT20230
C	ANT20235
CALI TIME ( TIM(5) )	ANT20240
CALI SOLVER (NMAX, RHS, A, C)	ANT20245
CALI TIME ( TIM(6) )	ANT20250
C	ANT20255
C FILL IN UCON, TCON --- PRINT RESULTS	ANT20260
C	ANT20265
DO 50 I = 1, NMAX	ANT20270
IF (LDC(I).EQ.1) GO TO 60	ANT20275
TCON(I) = FMI * A(I)	ANT20280

UCON(I) = (1./FMU) * BVAL(I)	ANT20285
GO TO 50	ANT20290
60 TCON(I) = BVAL(I)	ANT20295
UCON(I) = A(I)	ANT20300
50 CONTINUE	ANT20305
WRITE (6,2000) ITTL	ANT20310
IF (ISTRS.EQ.0) WRITE (6,2050)	ANT20315
IF (ISTRS.EQ.1) WRITE (6,2060)	ANT20320
WRITE (6,2100)	ANT20325
DO 70 I = 1,NSEG	ANT20330
J = I + NSEG	ANT20335
DO 80 N = 1,2	ANT20340
80 PXYZ(N) = (XYZ(1,1,N) + XYZ(I,2,N))/2.	ANT20345
70 WRITE (6,2200) I, UCON(I), UCON(J), TCON(I), TCON(J),	ANT20350
1 LHC(I), LHC(J), PXYZ(1), PXYZ(2)	ANT20355
IF (IPUNCH.EQ.0) RETURN	ANT20360
DO 120 I = 1,NSEG	ANT20365
J = I + NSEG	ANT20370
120 WRITE (7,2500) I, UCON(I), UCON(J)	ANT20375
DO 130 I = 1,NSEG	ANT20380
J = I + NSEG	ANT20385
130 WRITE (7,2500) I, TCON(I), TCON(J)	ANT20390
RETURN	ANT20395
2000 FORMAT (1H1, 16A5 // 10X 19HBOUNDARY CONDITIONS)	ANT20400
2050 FORMAT ( / 4( 18H PLANE STRAIN **** ) )	ANT20405
2060 FORMAT ( / 4( 18H PLANE STRESS **** ) )	ANT20410
2100 FORMAT (// 4X 4H SEG 7X 2H01 10X 2H02 10X 2HT1 10X 2HT2 8X 4HLC1	ANT20415
1 6X 4HLC2 8X 2HX1 10X 2HX2 //)	ANT20420
2200 FORMAT (2X I5, 2F12.8, 2F12.0, 6X I1, 11X I1, 2F12.6)	ANT20425
2300 FORMAT (1H1, 16A5 // 10X 22HRIGHT HAND SIDE VECTOR //)	ANT20430
2400 FORMAT (5X, I5, 2E15.0)	ANT20435
2500 FORMAT ( I10, 2E30.10)	ANT20440
3000 FORMAT (/// ( 2(8F12.6 /) //) )	ANT20445
END	ANT20450

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SUBROUTINE DELINT (G, RHS)
COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), ICON(200), LDC(200)
COMMON / ARRAY2 / BVAL(200)
COMMON / MATCON / PI, PMU, PQ(2,2), MU(2), FLEX(6), STIF(6), AV(2,2)
COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NHDYP
COMMON / CONTR2 / T1TL(16), IPUNCH, ISTRS, IRDY
DIMENSION A(2), E1(2), E2(2), P(2), X(2,2), R1(2), R2(2)
DIMENSION ISYM(2), G(NSIZE,NSIZE)
DIMENSION XX1(50), XX2(50), XX3(50), XX4(50)
DOUBLE PRECISION RHS(NSIZE)
COMPLEX PM, MU, AX
COMPLEX AK(2), BK(2), ZK1(2), ZK2(2), DI1(2), DI2(2)
COMPLEX U(2,2), T(2,2), TRANS, LOG1(2), LOG2(2)
COMPLEX MIKPOW
INTEGER SGN1
SPHT = 1.
DO 63 I = 1,50
XX1(I) = 0.
XX2(I) = 0.
XX3(I) = 0.
XX4(I) = 0.
63 CONTINUE
DO 10 I = 1,NSIZE
DO 10 J = 1,NSIZE
10 G(I,J) = 0.
DO 20 M = 1,NTOTAL
M1 = (M-1)/NSEG
M2 = M - M1*NSEG
M3 = M2 + NSEG
C
C COMPUTE SYMMETRY COEFFICIENTS USING Y, THEN X
C
IFLAG = 1
DO 16 K = 1,2
J = 3 - K
I = (M-1)/(NSEG*((2**J)/2))
ISYM(K) = (-1)**I
IF (I.EQ.0) ISYM(K) = 1
16 IFLAG = IFLAG * ISYM(K)
DO 30 J = 1,2
IF (IFLAG.GT.0) GO TO 25
X(1,J) = XY7(M2,2,J) * ISYM(J)
X(2,J) = XY7(M2,1,J) * ISYM(J)
GO TO 35
25 X(1,J) = XY7(M2,1,J) * ISYM(J)
X(2,J) = XY7(M2,2,J) * ISYM(J)
35 CONTINUE
C
C DEFINE DIRECTION OF THE LINE SEGMENT F2 = A(J) / AMAG
C
30 A(J) = X(2,J) - X(1,J)
AMAG = SQRT (A(1)**2 + A(2)**2)
DO 33 I = 1,2
E2(I) = A(I)/AMAG
J = 3 - I
33 E1(I) = E2(I) * (-1)**(J+1)

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 ANT25005  
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C CALCULATE THE ANGLES T1 AND T2 AND THE DISTANCE D

C  
 DO 20 N = 1,NSEG  
 DO 15 J = 1,2  
 P(J) = (XYZ(N,1,J) + XYZ(N,2,J))/2.  
 R1(J) = X(1,J) - P(J)  
 R2(J) = X(2,J) - P(J)  
 15 CONTINUE  
 CALL DOTPRD (R1, E1, U)  
 CALL DOTPRD (R1, E2, R12)  
 CALL DOTPRD (R2, E2, R22)  
 CALL DOTPRD (R1, R1, R1MAG)  
 CALL DOTPRD (R2, R2, R2MAG)  
 R1MAG = SQRT (R1MAG)  
 R2MAG = SQRT (R2MAG)  
 RA = ARS(R12)  
 RB = ARS(R22)  
 RMAG = AMAX1 (RA, RB)  
 IF (ABS(D / RMAG).LT.1.0E-03) GO TO 40

C  
 C CALCULATE D11, D12 FOR D.NE.0

C  
 TN1 = R12 / D  
 TN2 = R22 / D  
 DO 50 I = 1,2  
 AK(I) = F1(2)\*MU(I) + E1(1)  
 BK(I) = F1(1)\*MU(I) - E1(2)  
 ZK1(I) = D \* (AK(I) + BK(I) \* TN1)  
 ZK2(I) = D \* (AK(I) + BK(I) \* TN2)  
 LOG1(I) = CLOG( ZK1(I) )  
 LOG2(I) = CLOG( ZK2(I) )  
 DPHI = AJMAG (LOG2(I) - LOG1(I))  
 SPHI = SIGN (SPHI, DPHI)  
 IF (ABS(DPHI).GT.PI) LOG2(I) = LOG2(I) - SPHI \* CMPLX(0.0, 2.\*PI)  
 D11(I) = ZK2(I) \* (LOG2(I) - 1.) / BK(I)  
 1 - ZK1(I) \* (LOG1(I) - 1.) / BK(I)  
 D12(I) = (LOG2(I) - LOG1(I)) / BK(I)  
 50 CONTINUE  
 GO TO 60  
 40 CONTINUE

C  
 C CALCULATE D11, D12 FOR D.EQ.0

C  
 DO 55 I = 1,2  
 BK(I) = F1(1)\*MU(I) - E1(2)  
 D11(I) = R22 \* (CLOG(BK(I)\*R22) - 1.)  
 1 - R12 \* (CLOG(BK(I) \* R12) - 1.)  
 D12(I) = (ALOG(RH) - ALOG(RA)) / BK(I)  
 55 CONTINUE  
 60 CONTINUE

C  
 C CALCULATE DFLU, DELT INTEGRALS

C  
 DO 64 I = 1,2  
 DO 64 J = 1,2  
 T(J,I) = CM\_X(U,0,0,J)  
 U(J,I) = CMPLX(U,0,0,0)  
 64 CONTINUE

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 ANT25560  
 ANT25565  
 ANT25570

00 65 I = 1,2	ANT25575
00 65 J = 1,2	ANT25580
00 65 K = 1,2	ANT25585
SGNT = (-1)**(3-I)	ANT25590
MUKPOW = MU(K)**(2-I)	ANT25595
U(J,I) = U(J,I) + PQ(1,K)*AX(J,K)*DI1(K)	ANT25600
T(J,I) = T(J,I) + (MU(K)*E1(1)-E1(2))*MUKPOW*SGNT	ANT25605
1                   *AX(J,K)*DI2(K)	ANT25610
65 CONTINUE	ANT25615
XX1(N) = XX1(N) + 2.*REAL(T(1,1))	ANT25620
XX2(N) = XX2(N) + 2.*REAL(T(1,2))	ANT25625
XX3(N) = XX3(N) + 2.*REAL(T(2,1))	ANT25630
XX4(N) = XX4(N) + 2.*REAL(T(2,2))	ANT25635
75 00 85 IX = 1,2	ANT25640
00 85 JX = 1,2	ANT25645
N4 = N + (IX-1)*NSFG	ANT25650
M4 = M2 + (JX-1)*NSFG	ANT25655
IF (IX.EQ.JX.AND.M.FQ.N) T(IX,JX) = CMPLX(0.25,0.0)	ANT25660
IF (LDC(M4).EQ.1) GO TO 90	ANT25665
TRANS = U(IX,JX)	ANT25670
U(IX,JX) = -(1./FMU) * T(IX,JX)	ANT25675
T(IX,JX) = -FMU * TRANS	ANT25680
90 RHS(N4) = RHS(N4) + 2.*REAL(U(IX,JX)) * RVAL(M4) * ISYM(JX)	ANT25685
G(N4,M4) = G(N4,M4) + 2.*REAL(T(IX,JX)) * ISYM(JX)	ANT25690
85 CONTINUE	ANT25695
20 CONTINUE	ANT25700
WRITE(6,2500) TITL	ANT25705
2600 FORMAT(1H1,16A5//10X,17HCOLUMN SUM CHECKS//)	ANT25710
00 95 I = 1,NSEG	ANT25715
95 WRITE (6,2500) I, XX1(I), XX2(I), XX3(I), XX4(I)	ANT25720
RETURN	ANT25725
2000 FORMAT(1X,3I4,3F8.5,10F9.5 )	ANT25730
2100 FORMAT ( 5X I4, 2F20.10, I4)	ANT25735
2200 FORMAT(1X,3I4,12F9.4 )	ANT25740
2300 FORMAT (10X, 4E15.7)	ANT25745
2400 FORMAT (10X, 6I4, 6E15.7)	ANT25750
2500 FORMAT (3X, I4, 4F20.15)	ANT25755
END	ANT25760

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SUBROUTINE INSOLU( C )
COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), TCON(200), LDC(200)
COMMON / ARRAY3 / PIIN(100,2)
COMMON / MATCON / P1,FMU,PQ(2,2),MU(2),FLEX(6),STIF(6),AX(2,2)
COMMON / TIMERS / TIM (10)
COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NRDYP
COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, IRDY
DIMENSION C(100,6), A(4), PXYZ(3)
COMPLEX PN, MU, AX
IF (IRDY.NE.0) GO TO 100
110 IF (NPT.EQ.0) RETURN
CALL TIME ( TIM(7) )
WRITE (6,2000) TITL
IF (ISTRS.EQ.0) WRITE (6,2050)
IF (ISTRS.EQ.1) WRITE (6,2060)
C
C CALL FOR CALCULATION OF DELD AND DELS
C
WRITE (6,2100)
A(4) = 0.
CALL DFLSD (C)
DO 10 NP = 1,NPT
C(NP,5) = C(NP,5) * 2.
A(1) = STIF(1)*C(NP,4) + STIF(3)*C(NP,5) + STIF(2)*C(NP,6)
A(2) = STIF(3)*C(NP,4) + STIF(6)*C(NP,5) + STIF(5)*C(NP,6)
A(3) = STIF(2)*C(NP,4) + STIF(5)*C(NP,5) + STIF(4)*C(NP,6)
WRITE (6,2200) NP, (A(K),K=1,4), PTIN(NP,1), PTIN(NP,2)
10 CONTINUE
CALL TIME ( TIM(8) )
RETURN
100 WRITE (6,2000) TITL
DO 120 I = 1,NSIG
J = I + NSIG
120 READ (5,1100) N, UCON(I), UCON(J)
DO 130 I = 1,NSIG
J = I + NSIG
130 READ (5,1100) N, TCON(I), TCON(J)
WRITE (6,2300)
DO 140 I = 1,NSIG
J = I + NSIG
DO 150 N = 1,2
150 PXYZ(N) = (XYZ(I,1,N) + XYZ(I,2,N))/2.
140 WRITE (6,2400) I, UCON(I), UCON(J), TCON(I), TCON(J),
1 LDC(I), LDC(J), PXYZ(1), PXYZ(2)
GO TO 110
1100 FORMAT (I10, 2E30.10)
2000 FORMAT (1H1, 10X, 1HAs)
2050 FORMAT ( / 4( 18H PLANE STRAIN **** ) )
2060 FORMAT ( / 4( 18H PLANE STRESS **** ) )
2100 FORMAT (6HOPINI, 2X 10H SIGMA(AX) 2X 10H SIGMA(VY) 2X
1 10H SIGMA(VY) 2X 10H SIGMA(ZZ) 5X 2H X 6X 2H Y)
2200 FORMAT (2X 13, 2X 4F12.2, 2F8.4)
2300 FORMAT (// 4X 4H SIG 7X 2HUI 10X 2HUI2 10X 2HT1 10X 2HT2 4X 4H LDC1
1 6X 4H LDC2 4X 2HX1 10X 2HX2 //)
2400 FORMAT (2X 15, 2F12.8, 2F12.0, 6X I1, 11X I1, 2F12.6)
END

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SUBROUTINE DELSU (G)
COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), ICON(200), LUC(200)
COMMON / ARRAY3 / PTIN(100,2)
COMMON / MATCON / P1,FMU,PQ(2,2),MU(2),FLEX(6),STIF(6),AY(2,2)
COMMON / CONTR1 / NSEG, NSYM, NIOTAL, NSTZF, NPT, NHDYP
COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, IRDY
DIMENSION A(2), E1(2), E2(2), P(2), X(2,2), R1(2), R2(2)
DIMENSION ISYM(2), G(100,6)
COMPLEX PQ, MU, AX
COMPLEX AK(2), GK(2), ZK1(2), ZK2(2), D13(2), D14(2), DU, DS
COMPLEX LOG1(2), LOG2(2)
SPHT = 1.
DO 10 I = 1,100
DO 10 J = 1,6
10 G(I,J) = 0.
DO 20 M = 1,NIOTAL
M1 = (M-1)/NSEG
M2 = M - M1*NSEG
C
C COMPUTE SYMMETRY COEFFICIENTS USING Y, THEN X
IFLAG = 1
DO 16 K = 1,2
J = 3 - K
I = (M-1)/(NSEG*((2**J)/2))
ISYM(K) = (-1)**I
IF (I.EQ.0) ISYM(K) = 1
16 IFLAG = IFLAG * ISYM(K)
DO 32 J = 1,2
IF (IFLAG.GT.0) GO TO 23
X(1,J) = XY7(M2,2,J) * ISYM(J)
X(2,J) = XY7(M2,1,J) * ISYM(J)
GO TO 35
23 X(1,J) = XY7(M2,1,J) * ISYM(J)
X(2,J) = XY7(M2,2,J) * ISYM(J)
35 CONTINUE
C
C DEFINE DIRECTION OF THE LINE SEGMENT E2 = A(J)/AMAG
C
32 A(J) = X(2,J) - X(1,J)
AMAG = SQRT (A(1)**2 + A(2)**2)
DO 33 I = 1,2
E2(I) = A(I)/AMAG
J = 3 - I
33 E1(I) = E2(I) * (-1)**(J+1)
C
C CALCULATE THE ANGLES T1 AND T2 AND THE DISTANCE U
C
DO 20 N = 1,NPT
DO 15 J = 1,2
P(J) = PTIN(N,J)
R2(I) = X(2,J) - P(J)
R1(I) = X(1,J) - P(J)
15 CONTINUE
U1 = 0.
U2 = 0.
DO 17 J=1,2
U1 = U1 + R1(J)*R1(J)

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17	U2 = D2 + R2(J)*R2(J)	ANT35285
	D1 = SQRT(D1)	ANT35290
	D2 = SQRT(D2)	ANT35295
	CALI DOTPRD (R1, E1, J)	ANT35300
	CALI DOTPRD (R1, E2, R12)	ANT35305
	CALI DOTPRD (R2, E2, R22)	ANT35310
	CALI DOTPRD (R1, R1, R1MAG)	ANT35315
	CALI DOTPRD (R2, R2, R2MAG)	ANT35320
	R1MAG = SQRT (R1MAG)	ANT35325
	R2MAG = SQRT (R2MAG)	ANT35330
	RA = ABS(R12)	ANT35335
	RB = ABS(R22)	ANT35340
	RMAG = AMAX1(RA,RB)	ANT35345
	IF (ABS(D / RMAG).LT.1.0E-03) GO TO 40	ANT35350
	TN1 = R12 / D	ANT35355
	TN2 = R22 / D	ANT35360
C		ANT35365
C	CALCULATE D13, D14 FOR D.NE.0	ANT35370
C		ANT35375
	DO 45 I = 1,2	ANT35380
	AK(I) = F1(2)*MU(I) + E1(1)	ANT35385
	BK(I) = E1(1)*MU(I) - F1(2)	ANT35390
	ZK1(I) = D * (AK(I) + BK(I) * TN1)	ANT35395
	ZK2(I) = D * (AK(I) + BK(I) * TN2)	ANT35400
	LOG1(I) = CLOG( ZK1(I) )	ANT35405
	LOG2(I) = CLOG( ZK2(I) )	ANT35410
	DPHT = AIMAG (LOG2(I) - LOG1(I))	ANT35415
	SPHT = SIGN (SPHT, DPHT)	ANT35420
	IF (ABS(DPHT).GT.PI) LOG2(I) = LOG2(I) - SPHT * CMPLX(0.0, 2.*PI)	ANT35425
	D13(I) = (LOG2(I) - LOG1(I)) / BK(I)	ANT35430
	D14(I) = -(1./ZK2(I) - 1./ZK1(I) ) / BK(I)	ANT35435
45	CONTINUE	ANT35440
	GO TO 50	ANT35445
C		ANT35450
C	CALCULATE D13, D14 FOR D.EQ.0	ANT35455
C		ANT35460
40	CONTINUE	ANT35465
	DO 55 I = 1,2	ANT35470
	BK(I) = F1(1)*MU(I) - F1(2)	ANT35475
	D13(I) = (ALOG(RB) - ALOG(RA)) / BK(I)	ANT35480
	D14(I) = -(1./R22 - 1./R12) / BK(I)*2	ANT35485
55	CONTINUE	ANT35490
50	CONTINUE	ANT35495
C		ANT35500
C	CALCULATE DD, DS, STRAINS AND STRESSES	ANT35505
C		ANT35510
	L1 = 3	ANT35515
	DO 25 I = 1,2	ANT35520
	DO 25 L = 1,2	ANT35525
	L1 = L1 + 1	ANT35530
	DO 25 J = 1,2	ANT35535
	DD = CMPLX(0.0+0.0)	ANT35540
	DS = CMPLX(0.0+0.0)	ANT35545
	DO 30 K = 1,2	ANT35550
	DD = ((PD(J,K)*AX(I,K)+D13(K)*(MU(K)+*(L-1))	ANT35555
	+PD(J,K)*AX(L,K)+D13(K)*(MU(K)+*(I-1))) / 2. + DD	ANT35560
	DS = (-((BK(K)+E1(1)-F1(2))*MU(K)+*(2-J)+BX(I,K)+BX(L,K)+*(L-1)+D14(K)	ANT35565
	+((-1)+*(3-J))	ANT35570

2	-(M1(K)*E1(1)-F1(2))*MU(K)**(2-J)*AX(L,K)*M1(K)**(I-1)*DI4(K)	ANI35575
3	*((-1)**(3-J))/2. + US	ANI35580
30	CONTINUE	ANI35585
	M4 = M2 + (J-1)*NSE6	ANI35590
	G(N,L1) = G(N,L1) - 2.*REAL(DU)*TCON(M4)*ISYM(J)	ANI35595
1	+ 2.*REAL(DS)*UCON(M4)*ISYM(J)	ANI35600
25	CONTINUE	ANI35605
20	CONTINUE	ANI35610
	RETURN	ANI35615
	END	ANI35620

SUBROUTINE HDYSR (6)

COMMON / ARRAY1 / XYZ(100,2,2), UCON(200), TCON(200), LDC(200)

COMMON / ARRAY4 / NHDY(50,3)

COMMON / MATCON / PI,FMU,PQ(2,2),MU(2),FLEX(6),STIF(6),AX(2,2)

COMMON / CONTR1 / NSEG, NSYM, NTOTAL, NSIZE, NPT, NBDYP

COMMON / CONTR2 / TITL(16), IPUNCH, ISTRS, IRDY

COMMON / TIMERS / TIM(10)

DIMENSION A(2), E1(2), E2(2), P(3,2), R(2), DU(2), G(50,6)

DIMENSION Q(3,3), E(3,3), COFAC(3,3), T(3,3), T1(3,3)

DIMENSION RHS(3), TRAC(2), TCN(2), TEMP(3,3), ANS(3)

COMPLEX PQ, MU, AX

IF (NBDYP.EQ.0) RETURN

CALL TIME ( TIM(9) )

WRITE (6,2000) TITL

WRITE (6,2100) ((NBDY(I,J),J=1,3),I=1,NBDYP)

WRITE (6,2000) TITL

IF (ISTRS.EQ.0) WRITE (6,2050)

IF (ISTRS.EQ.1) WRITE (6,2060)

WRITE (6,2200)

C

C I0 = BASE SEGMENT NUMBER

C I1 = REAR DIFFERENCE SEGMENT NUMBER

C I2 = FORWARD DIFFERENCE SEGMENT NUMBER

C

DO 15 N = 1,NBDYP

I0 = NHDY (N,1)

I1 = NHDY (N,2)

I2 = NHDY (N,3)

DO 20 M = 1,2

P(1,M) = (XYZ(I0,1,M) + XYZ(I0,2,M))/2.

P(2,M) = (XYZ(I1,1,M) + XYZ(I1,2,M))/2.

P(3,M) = (XYZ(I2,1,M) + XYZ(I2,2,M))/2.

R(M) = P(3,M) - P(2,M)

20 A(M) = XYZ(I0,2,M) - XYZ(I0,1,M)

SMAG = SQRT(R(1)\*\*2 + R(2)\*\*2)

AMAG = SQRT(A(1)\*\*2 + A(2)\*\*2)

C

C CALCULATE DII/DS, TRAC FOR GLOBAL COORDINATE SYSTEM

C

DO 25 M = 1,2

E2(M) = A(M)/ AMAG

K = 3 - M

E1(K) = E2(M) \* (-1)\*\*(K+1)

I3 = I1 + (M-1)\*NSEG

I4 = I2 + (M-1)\*NSEG

I5 = I0 + (M-1)\*NSEG

DU(M) = (UCON(I4) - UCON(I3))/SMAG

25 TRAC(M) = TCON(I5)

C

C TRANSFORM DII/DS INTO EPS --- TRAC INTO TCN, IN LOCAL COORDINATES

C

CALL DOTPRD (DU, E2, EPS)

DO 40 I = 1,2

40 TCN(I) = 0.

DO 45 I = 1,2

TCN(1) = TCN(1) + E1(1)\*TRAC(1)

45 TCN(2) = TCN(2) + E2(1)\*TRAC(1)

ANT40000

ANT40005

ANT40010

ANT40015

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ANT40065

ANT40070

ANT40075

ANT40080

ANT40085

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ANT40095

ANT40100

ANT40105

ANT40110

ANT40115

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ANT40125

ANT40130

ANT40135

ANT40140

ANT40145

ANT40150

ANT40155

ANT40160

ANT40165

ANT40170

ANT40175

ANT40180

ANT40185

ANT40190

ANT40195

ANT40200

ANT40205

ANT40210

ANT40215

ANT40220

ANT40225

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ANT40280

C  
C CALCULATE TRANSFORMATION MATRIX T(I,J) AND ITS INVERSE TI(I,J)  
C

C = E1(1)  
S = E1(2)  
T(1,1) = C\*C  
T(1,2) = S\*S  
T(1,3) = S\*C\*2.  
T(2,1) = S\*S  
T(2,2) = C\*C  
T(2,3) = -S\*C\*2.  
T(3,1) = -S\*C  
T(3,2) = S\*C  
T(3,3) = C\*C - S\*S

C  
C \*\*\*\*\* \*\*:  
C

TI(1,1) = T(1,1)  
TI(1,2) = T(1,2)  
TI(1,3) = -T(1,3)  
TI(2,1) = T(2,1)  
TI(2,2) = T(2,2)  
TI(2,3) = -T(2,3)  
TI(3,1) = -T(3,1)  
TI(3,2) = -T(3,2)  
TI(3,3) = T(3,3)

C  
C CALCULATE MATERIAL STIFFNESSES IN LOCAL COORDINATE SYSTEM  
C

K = 0  
DO 50 I = 1,3  
DO 50 J = 1,3  
K = K + 1  
Q(I,J) = STIF(K) \* (2.\*\*(J/3))  
IF (I.EQ.J) GO TO 50  
Q(J,I) = STIF(K)  
50 CONTINUE  
DO 55 I = 1,3  
DO 55 J = 1,3  
TEMP(I,J) = 0.  
DO 55 K = 1,3  
55 TEMP(I,J) = TEMP(I,J) + Q(I,K)\*TI(K,J)  
DO 60 I = 1,3  
DO 60 J = 1,3  
Q(I,J) = 0.  
DO 60 K = 1,3  
60 Q(I,J) = Q(I,J) + T(I,K)\*TEMP(K,J)

C  
C CALCULATE COEFFICIENTS OF REARRANGED EQUATIONS  
C

E(1,1) = Q(1,1) - Q(1,2)\*Q(1,2)/Q(2,2)  
E(1,2) = Q(1,2)  
E(1,3) = Q(1,3) - Q(1,2)\*Q(2,3)/Q(2,2)  
E(2,1) = -Q(1,2)  
E(2,2) = Q(2,2)  
E(2,3) = -Q(2,3)  
E(3,1) = Q(1,3) - Q(1,2)\*Q(2,3)/Q(2,2)  
E(3,2) = Q(2,3)

Q(1,3) = Q(1,3) /2.  
Q(2,3) = Q(2,3) /2.  
Q(3,3) = Q(3,3) /2.

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ANT40555  
ANT40560  
ANT40565  
ANT40570



E(3,3) = Q(3,3) - Q(2,3)*Q(2,3)/Q(2,2)	ANTI40575
DET = (F(1,1)*E(2,2)*E(3,3)) + (E(1,2)*E(2,3)*E(3,1)) +	ANTI40580
1 (F(1,3)*E(2,1)*E(3,2)) - (E(3,1)*E(2,2)*E(1,3)) -	ANTI40585
2 (F(3,2)*E(2,3)*E(1,1)) - (E(3,3)*F(2,1)*E(1,2))	ANTI40590
COFAC(1,1) = (E(2,2)*E(3,3)) - (E(3,2)*E(2,3))	ANTI40595
COFAC(1,2) = -1.0*(E(2,1)*E(3,3)) - (E(3,1)*E(2,3))	ANTI40600
COFAC(1,3) = (E(2,1)*E(3,2)) - (E(3,1)*E(2,2))	ANTI40605
COFAC(2,1) = -1.0*(E(1,2)*E(3,3)) - (E(3,2)*E(1,3))	ANTI40610
COFAC(2,2) = (E(1,1)*E(3,3)) - (E(3,1)*E(1,3))	ANTI40615
COFAC(2,3) = -1.0*(E(1,1)*E(3,2)) - (E(3,1)*E(1,2))	ANTI40620
COFAC(3,1) = (E(1,2)*E(2,3)) - (E(2,2)*E(1,3))	ANTI40625
COFAC(3,2) = -1.0*(E(1,1)*E(2,3)) - (E(2,1)*E(1,3))	ANTI40630
COFAC(3,3) = (E(1,1)*E(2,2)) - (E(2,1)*E(1,2))	ANTI40635
DO 65 I = 1,3	ANTI40640
DO 65 J = 1,3	ANTI40645
65 TEMP(I,J) = COFAC(J,I)/DET	ANTI40650
RHS(1) = TCN(1)	ANTI40655
RHS(2) = EPS*Q(2,2)	ANTI40660
RHS(3) = TCN(2)	ANTI40665
C	ANTI40670
C CALCULATE UNKNOWN HOOP STRESS	ANTI40675
C	ANTI40680
DO 70 I = 1,3	ANTI40685
ANS(I) = 0.	ANTI40690
DO 70 J = 1,3	ANTI40695
70 ANS(I) = ANS(I) + TEMP(I,J)*RHS(J)	ANTI40700
G(N,1) = TCN(1)	ANTI40705
G(N,2) = TCN(2)	ANTI40710
G(N,3) = ANS(2) * Q(2,2)	ANTI40715
G(N,4) = EPS	ANTI40720
G(N,5) = ANS(1)	ANTI40725
G(N,6) = ANS(3)	ANTI40730
WRITE (6,100) STZF, Q, T, TI, RHS, ANS	ANTI40735
100 FORMAT ( // (3(3E12.7 //)) )	ANTI40740
15 WRITE (6,2300) 10, (G(N,M),M=1,4),P(1,1),P(1,2)	ANTI40745
CAL TIME ( TIM(10) )	ANTI40750
RETURN	ANTI40755
1000 FORMAT (24I3)	ANTI40760
2000 FORMAT (1H1, 10X, 16A5)	ANTI40765
2050 FORMAT ( / 4( 18H PLANE STRAIN **** ) )	ANTI40770
2060 FORMAT ( / 4( 18H PLANE STRESS, **** ) )	ANTI40775
2100 FORMAT (/ 5X 11HBASE NUMBER 2X 11HREF NUMBER 3X 10HFWO NUMBER //	ANTI40780
1 ( 3I12 / ) )	ANTI40785
2200 FORMAT (7HSEGMENT 2X 10H NORMAL ST 2X 10H SHEAR STR 2X	ANTI40790
1 10H HOOP STRS 2X 10H HOOP STRN 5X 2H X 6X 2H Y)	ANTI40795
2300 FORMAT (2X I3, 2X 3F12.2, F12.9, 2F4.4)	ANTI40800
END	ANTI40805

```
SUBROUTINE DOTPRD (A, B, C)
  DIMENSION A(2), B(2)
  C = A(1)*B(1) + A(2)*B(2)
  RETURN
END
```

```
ANT45000
ANT45005
ANT45010
ANT45015
ANT45020
```

SUBROUTINE SOLVER (N, X, F, A)	AN150000
DIMENSION A(N,N), X(N), F(N), XX(160)	AN150005
DOUBLE PRECISION X	AN150010
DO 10 I = 1, N	AN150015
F(I) = 0.0	AN150020
10 CONTINUE	AN150025
N1 = N - 1	AN150030
DO 40 I = 2, N	AN150035
DO 45 J = I, N	AN150040
IF (ABS(A(I-1,I-1)) .GT. 0.) GO TO 45	AN150045
I1 = I - 1	AN150050
WRITE (6,510) I1	AN150055
RETURN	AN150060
45 CONTINUE	AN150065
CX = A(J,I-1) / A(I-1,I-1)	AN150070
K2 = I	AN150075
DO 50 K = I, N	AN150080
A(J,K2) = A(J,K2) - CX * A(I-1,K2)	AN150085
K2 = K2 + 1	AN150090
50 CONTINUE	AN150095
A(J,I-1) = CX	AN150100
55 CONTINUE	AN150105
60 CONTINUE	AN150110
C FORWARD PASS - OPERATE ON RIGHT HAND SIDE AS	AN150115
C ON MATRIX	AN150120
62 CONTINUE	AN150125
DO 70 I = 2, N	AN150130
DO 65 J = I, N	AN150135
X(J) = X(J) - X(I-1) * A(J,I-1)	AN150140
65 CONTINUE	AN150145
70 CONTINUE	AN150150
C	AN150155
C BACKWARD PASS - SOLVE FOR AX = B	AN150160
XX(N) = X(N) / A(N,N)	AN150165
DO 80 I = 1, N1	AN150170
SUM = 0.0	AN150175
I2 = N - I + 1	AN150180
DO 75 J = I2, N	AN150185
SUM = SUM + A(I2-1,J) * XX(J)	AN150190
75 CONTINUE	AN150195
XX(I2-1) = (X(I2-1)-SUM) / A(I2-1,I2-1)	AN150200
80 CONTINUE	AN150205
DO 90 I = 1, N	AN150210
F(I) = F(I) + XX(1)	AN150215
90 CONTINUE	AN150220
RETURN	AN150225
510 FORMAT(/1X 25HERROR RETURN FROM SEASOV I10,	AN150230
1 35HDIAGONAL TERM REDUCED TO ZERO / )	AN150235
END	AN150240

### 5.3 *EXAMPLE SOLUTIONS FOR ISOTROPIC AND ANISOTROPIC BOUNDARY-INTEGRAL EQUATION METHOD*

#### 5.3.1 *Tension of an Isotropic Plate*

The group of problems discussed herein are provided for two major purposes, the first to determine solution accuracy, and the second to provide guidelines in the use of the program, called DIPOME.

Since both tractions and displacements are assumed constant along each segment, it is logical to theorize that the solution will be more accurate for shorter segment lengths. If then two questions remain: What accuracy is obtainable, and how is this accuracy related to the segment length used in the model.

##### 5.3.1.1 *Circular Cutout*

A circular cutout, of unit radius, was modeled by segments of equal length. Ten problems were solved, with the only variable being the number of segments employed. In each problem the stress distribution along the x and y axes interior to the plate was obtained. Stresses were computed at points ranging from 0.001 inches to over 5.0 inches from the surface of the cutout. Table 1 shows a comparison of the solutions obtained in three of these problems to the theoretical results of Timoshenko. These solutions follow the theoretical curve closely in all cases, except in the immediate vicinity of the cutout.

This behavior is due largely to the presence of a sharp corner at the intersection of each axis with the cutout, as shown in Figure 1. This is a consequence of the approximation of the surface by straight line segments. Use of shorter segment lengths reduces the sharpness of this corner and produces less distortion, as seen in the table. Further deviation from the theoretical solution is a result of the surface approximations inherent in averaging tractions and displacements over each segment.

Stresses at the surface of the cutout are computed by a finite difference technique, using the displacements of the two segments adjacent to each of the intersections, as shown in Figure 1. Once the strain is computed by the equation below, Hooke's law is used to obtain the stress. Surface stresses are computed at nodes around the entire cutout, with an average error of about two per cent. For brevity only the stress at the intersection of the cutout and y-axis is shown here (Table 1).

$$\epsilon_x = \frac{U_{x(2)} - U_{x(1)}}{L}$$

The influence of segment length on solution accuracy is summarized in Figure 3. The graph results from comparisons of interior stresses, where  $\gamma^*$  represents the last data point obtained before the data diverges from theoretical curve of Timoshenko.

#### 5.3.1.2 *Elliptical Cutout*

The problem of an elliptical cutout in an infinite plate under tension was solved by Inglis in 1913. He found that the maximum stresses in the plate occur at the surface of the cutout, at the point where the radius of curvature is smallest. The stress concentration here is given by:

$$SCF = 1 + 2 a/b$$

where  $a/b$  is the aspect ratio of the ellipse.

Prediction capability for a range of stress concentrations was investigated, and results are reported here for concentrations of 5, 10, and 40. By the equation above, aspect ratios of the resulting ellipses were 2.0, 4.5, and 19.5.

For each aspect ratio a number of problems was solved, using varying numbers of segments to model the elliptical surface. Elliptic coordinates were used to divide the surface, so that a constant value of segment length/radius of curvature was obtained. It may be shown that this will be accomplished by using equal increments of the coordinate  $\eta$ . It was hoped that accuracy of the internal solution might be related to this ratio.

The stress distribution along the  $x$  and  $y$  axes interior to the plate was obtained, at points ranging from 0.001 inch to 5.0 inches from the cutout surface. Tables 2 through 7 show a comparison of the results with the theoretical results of Inglis. Again it must be noted that the computed results are inaccurate for points very near the cutout surface, due to the sharp corner produced by the model (See Figure 2).

Stresses at the surface of the cutout are computed by a finite difference technique, as previously described. Results of this calculation are shown only at the intersection of the  $x$  and  $y$  axes with the cutout, and appear in Tables 2 through 7.

The relationship between interior solution accuracy and the value of segment length/radius of curvature employed in a given problem is shown in Figure 4. Results were obtained for four aspect ratios, and the plots are nearly straight lines for each aspect ratio, for values of length parameter down to 0.052. It may be seen that solution accuracy is functionally related to the ratio of segment length to radius of curvature, but this parameter alone does not characterize accuracy.

We see that the Boundary-Integral Equation method is a reliable numerical technique for the prediction of stress concentrations in two dimensional isotropic problems. Results indicate that the method is consistent as well as accurate in calculating stress concentrations as high as 40. Solution accuracy is dependent both on the stress concentration gradients present and on the length of segment used to model the surface.

In employing this program, it should be noted that solution time required for interior points is approximately ten times that required for boundary solution points.

#### *5.3.2 Tension of an Anisotropic Plate with a Circular Cutout*

The program used for the solution of the following problems is called ANISOT, and provides a solution capability for two dimensional generally anisotropic materials. The use of the program is restricted only by the requirement that the material employed be mid-plane symmetric. It is expected that the program will be especially useful in analysis of advanced fiber composites, so the problems solved here considered plates of boron-epoxy.

##### *5.3.2.1 Orthotropic Material*

A series of problems was solved, with the cutout surface represented by varying numbers of segments. In each case segments of equal length were used, and the number of segments ranged from 20 to 180. Identical series of problems were solved for plates of unidirectional boron-epoxy, of zero degree and ninety degree orientations.

Again the stress distribution along the x and y axes interior to the body was obtained, as well as surface stresses around the entire cutout.

The hoop stresses around the cutout at the surface were compared to the theoretical results of Lekhnitskii, and results appear in Tables 8 and 9. These stresses are computed directly from displacements and tractions, and thus provide a means of evaluating the boundary solution capability of the program. Results compared extremely well with the theoretical calculations, even for the higher stress concentrations.

The accuracy of the solutions obtained are dependent on both the stress concentration gradients present and length of segment used in the model. This behavior is expected, since the basic algorithms employed are similar to those of the isotropic program, DIPOME. Time required for interior solution points was again about ten times that for boundary points.

#### *5.3.3.2 Anisotropic Material*

The problem of a circular cutout in an infinite plate was next solved for a completely anisotropic material, unidirectional boron-epoxy at an orientation of 45 degrees. There was no symmetry about either the x or y axis, as had been present before, so the entire cutout surface was modeled.

As before, the hoop stresses at the surface of the cutout were obtained and are compared with the results of Lekhnitskii in Tables 10 and 11. Two problems were solved, one employing 20 segments, and the other 90 segments, to represent the surface. Here again agreement with theoretical results was excellent along the entire surface.



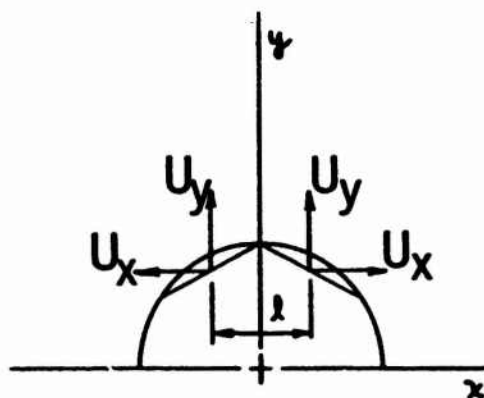


Figure 1: Dipole Model - Circle

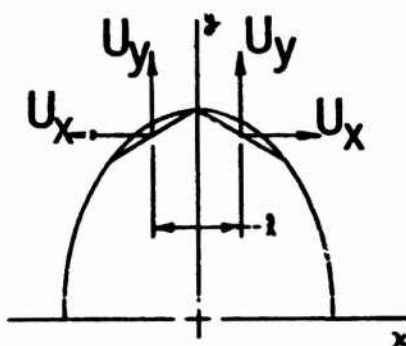


Figure 2: - Dipole Model - Ellipse

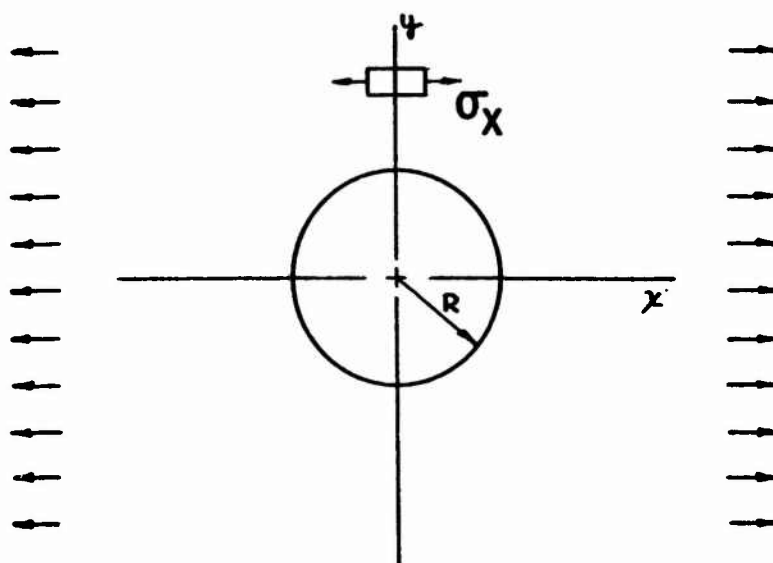


TABLE 1 — INTERIOR STRESS SOLUTIONS — DIPOME

Y - R	Timoshenko	40 Segments	120 Segments	200 Segments
0.00	3.00	3.02	3.02	3.09
.001	2.99	27.12	10.36	7.05
.005	2.96	7.03	3.78	3.21
.010	2.93	4.53	3.05	2.88
.020	2.87	3.31	2.80	2.83
.030	2.80	2.92	2.76	2.79
.040	2.74	2.74	2.72	2.74
.050	2.69	2.64	2.68	2.69
.070	2.58	2.52	2.58	2.58
.100	2.44	2.40	2.44	2.44
.200	2.07	2.07	2.07	2.07
.400	1.65	1.65	1.65	1.65
.600	1.42	1.42	1.42	1.42
.900	1.25	1.25	1.25	1.25
1.50	1.12	1.12	1.12	1.12
2.00	1.07	1.08	1.07	1.07
3.00	1.04	1.04	1.04	1.04

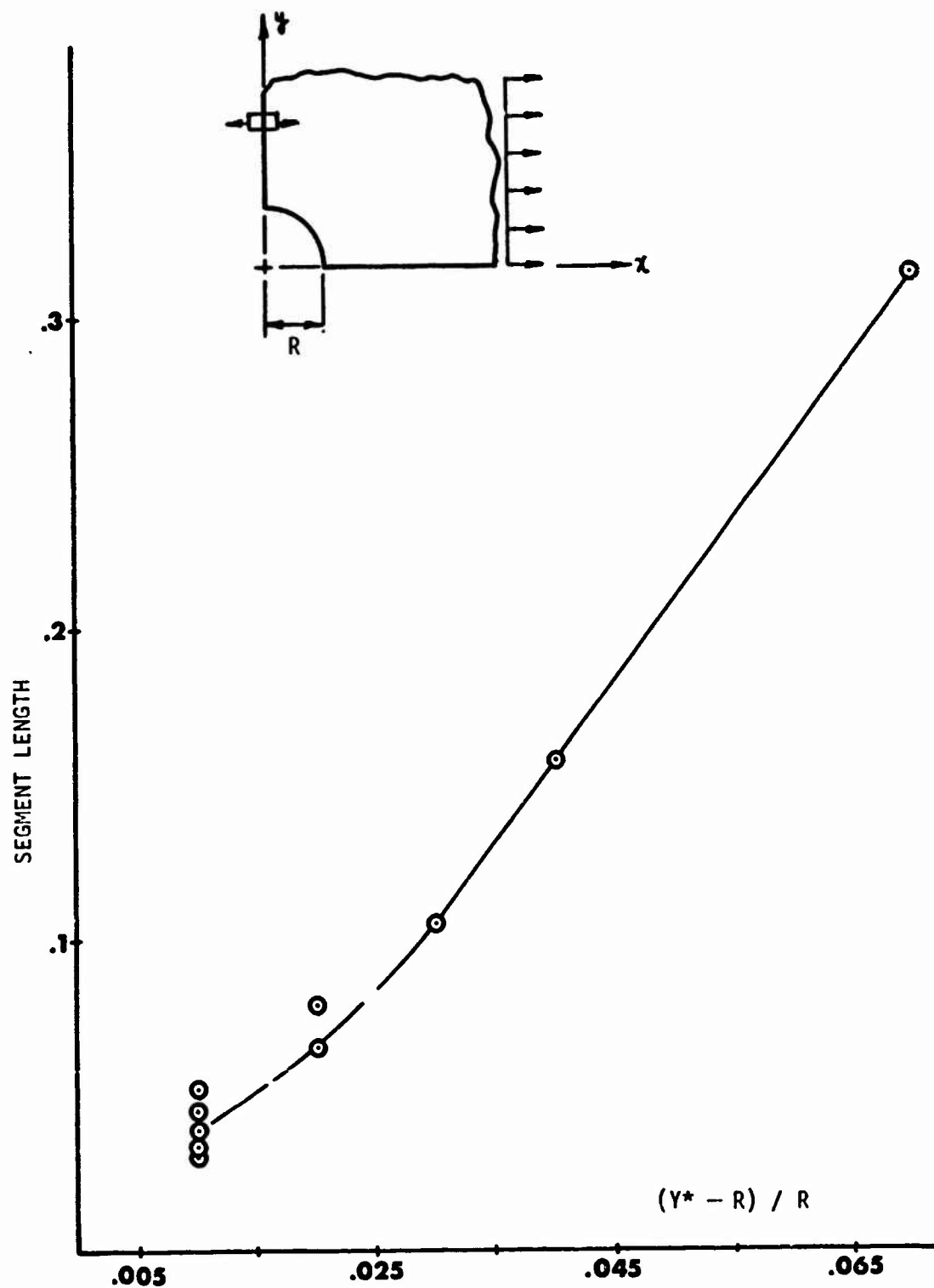
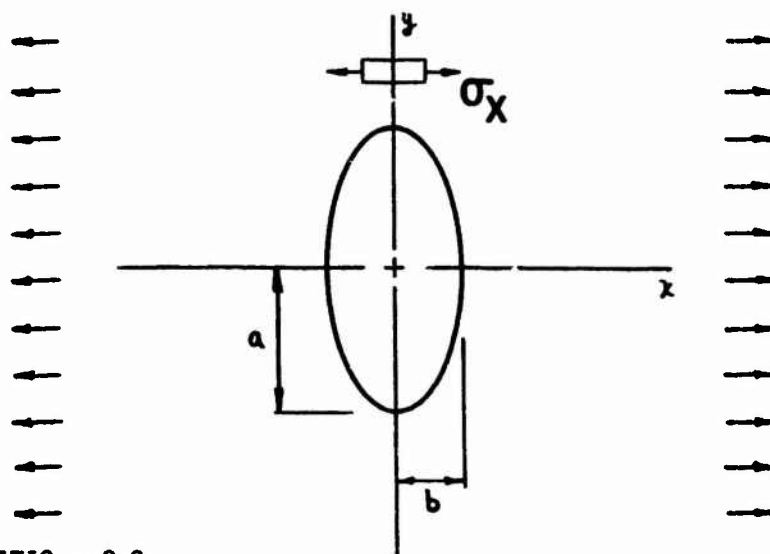


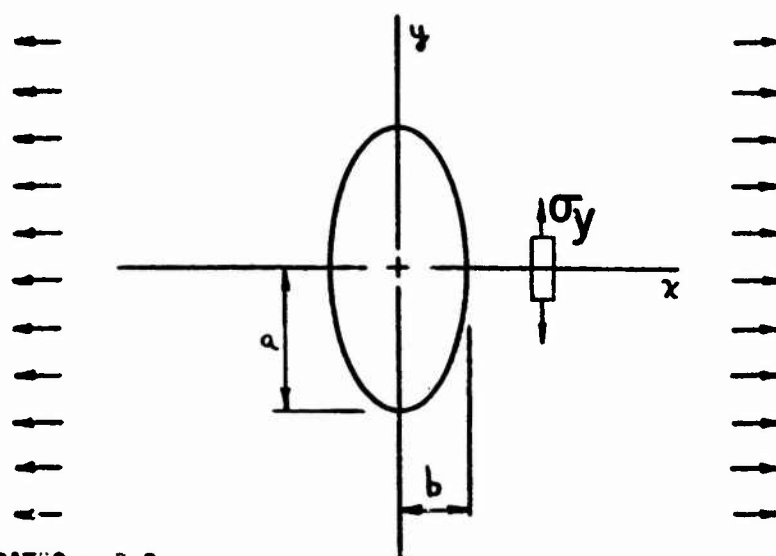
Figure 3: Accuracy of DIPOME - Circular Cutout



ASPECT RATIO = 2.0

TABLE 2 — INTERIOR STRESS SOLUTIONS — DIPOME

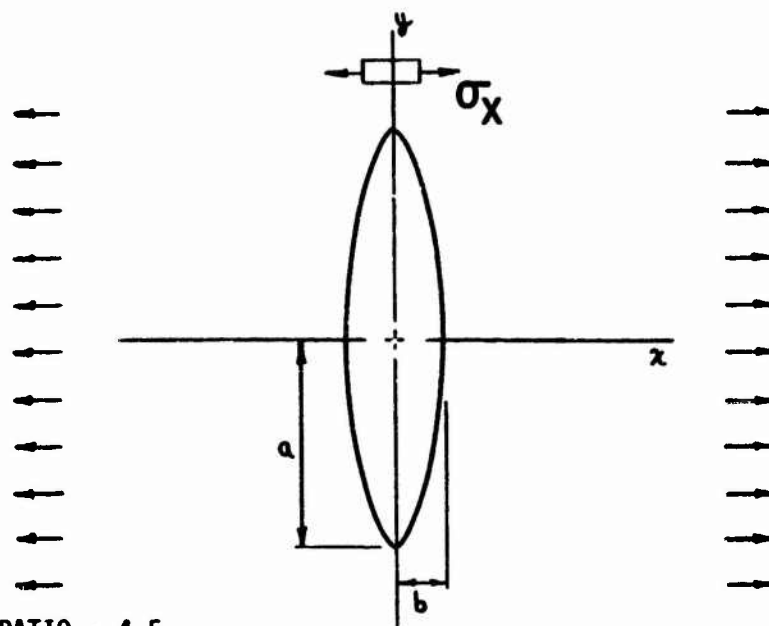
$\frac{y-a}{c}$	Ingilis	40 Segments	120 Segments	200 Segments
0.00	5.00	5.116	5.01	5.002
.001	4.96	101.9	35.2	22.3
.005	4.82	9.81	5.52	4.89
.010	4.65	6.13	4.59	4.56
.020	4.34	4.54	4.29	4.35
.030	4.03	4.04	4.08	4.09
.040	3.85	3.78	3.86	3.86
.050	3.65	3.59	3.66	3.66
.070	3.31	3.29	3.32	3.32
.100	2.92	2.92	2.93	2.93
.200	2.18	2.19	2.19	2.19
.400	1.61	1.62	1.62	1.62
.600	1.40	1.40	1.40	1.40
.900	1.24	1.24	1.24	1.24



ASPECT RATIO = 2.0

TABLE 3 - INTERIOR STRESS SOLUTIONS - DIPOME

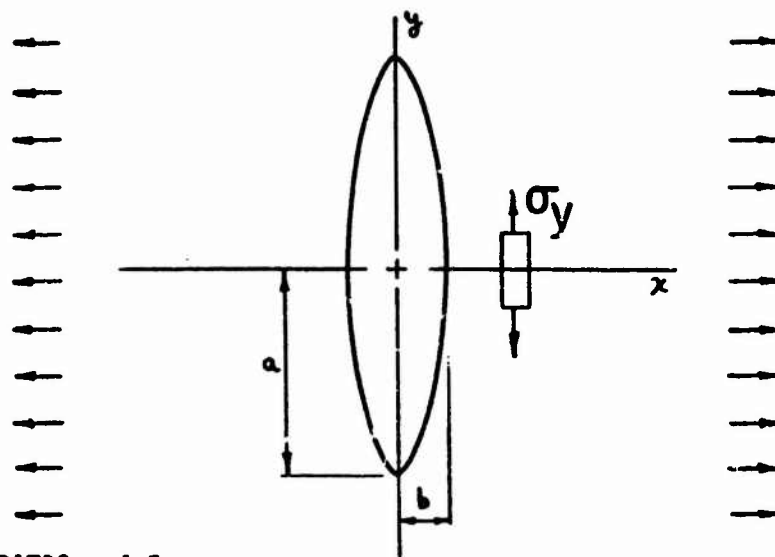
$\frac{x-b}{c}$	Inglis	40 Segments	120 Segments	200 Segments
0.00	-1.00	-0.983	-0.994	-0.996
.001	-0.997	-13.0	-4.90	-3.24
.005	-0.985	-2.42	-1.32	-1.11
.010	-0.970	-1.52	-1.03	-0.965
.020	-0.941	-1.09	-0.925	-0.926
.030	-0.912	-0.959	-0.895	-0.906
.040	-0.884	-0.888	-0.873	-0.882
.050	-0.857	-0.843	-0.850	-0.856
.070	-0.805	-0.782	-0.802	-0.804
.100	-0.805	-0.713	-0.729	-0.731
.200	-0.525	-0.518	-0.523	-0.524
.400	-0.251	-0.247	-0.250	-0.250
.600	-0.099	-0.097	-0.099	-0.099
.900	+0.006	+0.008	+0.007	+0.007



ASPECT RATIO = 4.5

TABLE 4 — INTERIOR STRESS SOLUTIONS — DIPOME

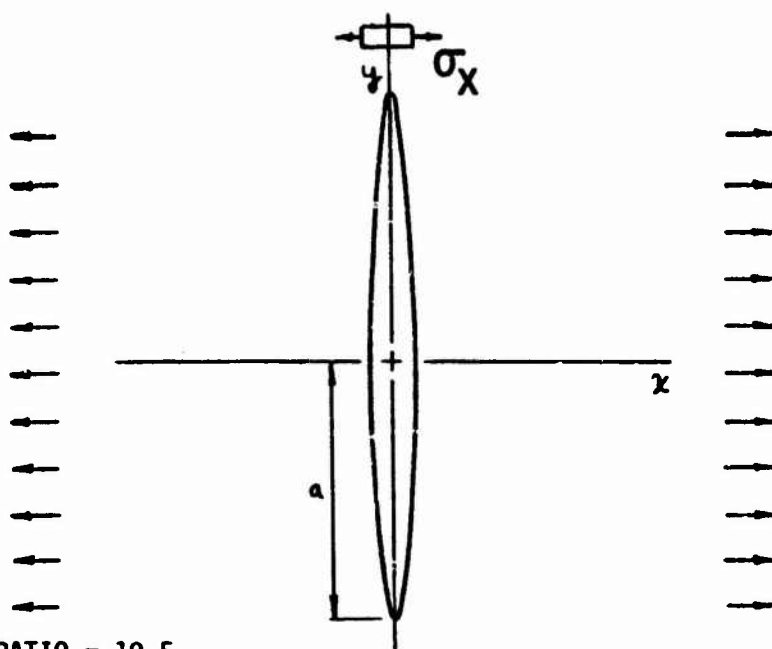
$\frac{y-a}{c}$	Inglis	40 Segments	120 Segments	200 Segments
0.00	10.0	11.33	10.14	10.04
.001	9.60	87.5	29.8	20.2
.005	8.31	11.37	8.30	8.32
.010	7.15	7.69	7.21	7.27
.020	5.67	5.75	5.75	5.75
.030	4.77	4.85	4.82	4.82
.040	4.16	4.24	4.20	4.20
.050	3.73	3.80	3.75	3.75
.070	3.14	3.19	3.16	3.16
.100	2.63	2.66	2.63	2.63
.200	1.90	1.91	1.90	1.90
.400	1.46	1.46	1.46	1.46
.600	1.30	1.30	1.30	1.30
.900	1.19	1.18	1.19	1.19



ASPECT RATIO = 4.5

TABLE 5 — INTERIOR STRESS SOLUTIONS — DIPOME

$\frac{x-b}{c}$	Inglis	40 Segments	120 Segments	200 Segments
0.00	-1.00	-0.984	-0.994	-0.997
.001	-0.997	-0.997	-0.994	-0.997
.005	-0.997	-2.43	-1.32	-1.11
.010	-0.974	-1.48	-1.03	-0.966
.020	-0.949	-1.08	-0.931	-0.937
.030	-0.923	-0.952	-0.908	-0.920
.040	-0.898	-0.893	-0.890	-0.893
.050	-0.874	-0.855	-0.870	-0.874
.070	-0.826	-0.804	-0.825	-0.826
.100	-0.758	-0.743	-0.756	-0.758
.200	-0.553	-0.549	-0.552	-0.553
.400	-0.258	-0.258	-0.258	-0.255
.600	-0.087	-0.088	-0.087	-0.087
.900	+0.029	+0.028	+0.029	+0.029

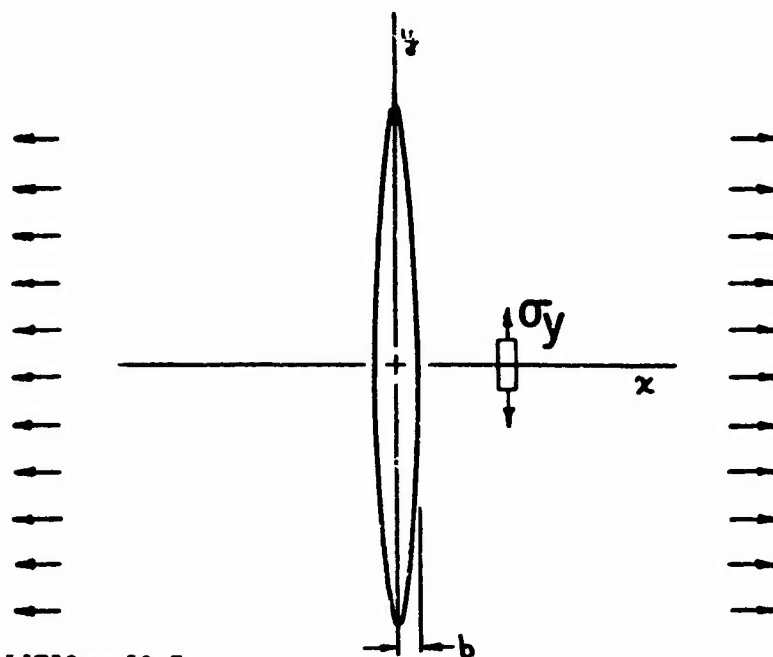


ASPECT RATIO = 19.5

TABLE 6 — INTERIOR STRESS SOLUTIONS — DIPOME

$\frac{y-a}{c}$	Inglis	40 Segments	120 Segments	200 Segments
0.00	40.0	88.7	50.9	44.5
.001	23.5	88.2	31.0	27.0
.005	10.9	16.9	11.7	11.4
.010	7.50	10.0	7.80	7.67
.020	5.22	6.28	5.32	5.27
.030	4.26	4.88	4.30	4.28
.040	3.70	4.12	3.72	3.71
.050	3.32	3.63	3.33	3.32
.070	2.83	3.03	2.84	2.83
.100	2.41	2.53	2.41	2.41
.200	1.81	1.85	1.81	1.81
.400	1.43	1.44	1.43	1.42
.600	1.28	1.29	1.28	1.28
.900	1.18	1.18	1.18	1.18





ASPECT RATIO = 19.5

TABLE 7 — INTERIOR STRESS SOLUTIONS — DIPOME

$\frac{x-b}{c}$	Inglis	40 Segments	120 Segments	200 Segments
0.00	-1.00	-.966	-.985	-.994
.001	-0.997	-10.9	-4.26	-2.89
.005	-0.989	-2.11	-1.22	-1.06
.010	-0.979	-1.37	-0.996	-0.959
.020	-0.957	-1.035	-0.921	-0.939
.030	-0.936	-0.928	-0.904	-0.926
.040	-0.915	-0.876	-0.890	-0.907
.050	-0.894	-0.845	-0.873	-0.887
.070	-0.852	-0.803	-0.834	-0.845
.100	-0.790	-0.752	-0.775	-0.784
.200	-0.598	-0.581	-0.587	-0.594
.400	-0.293	-0.294	-0.289	-0.291
.600	-0.099	-0.107	-0.099	-0.099
.900	+0.037	+0.030	+0.035	+0.037

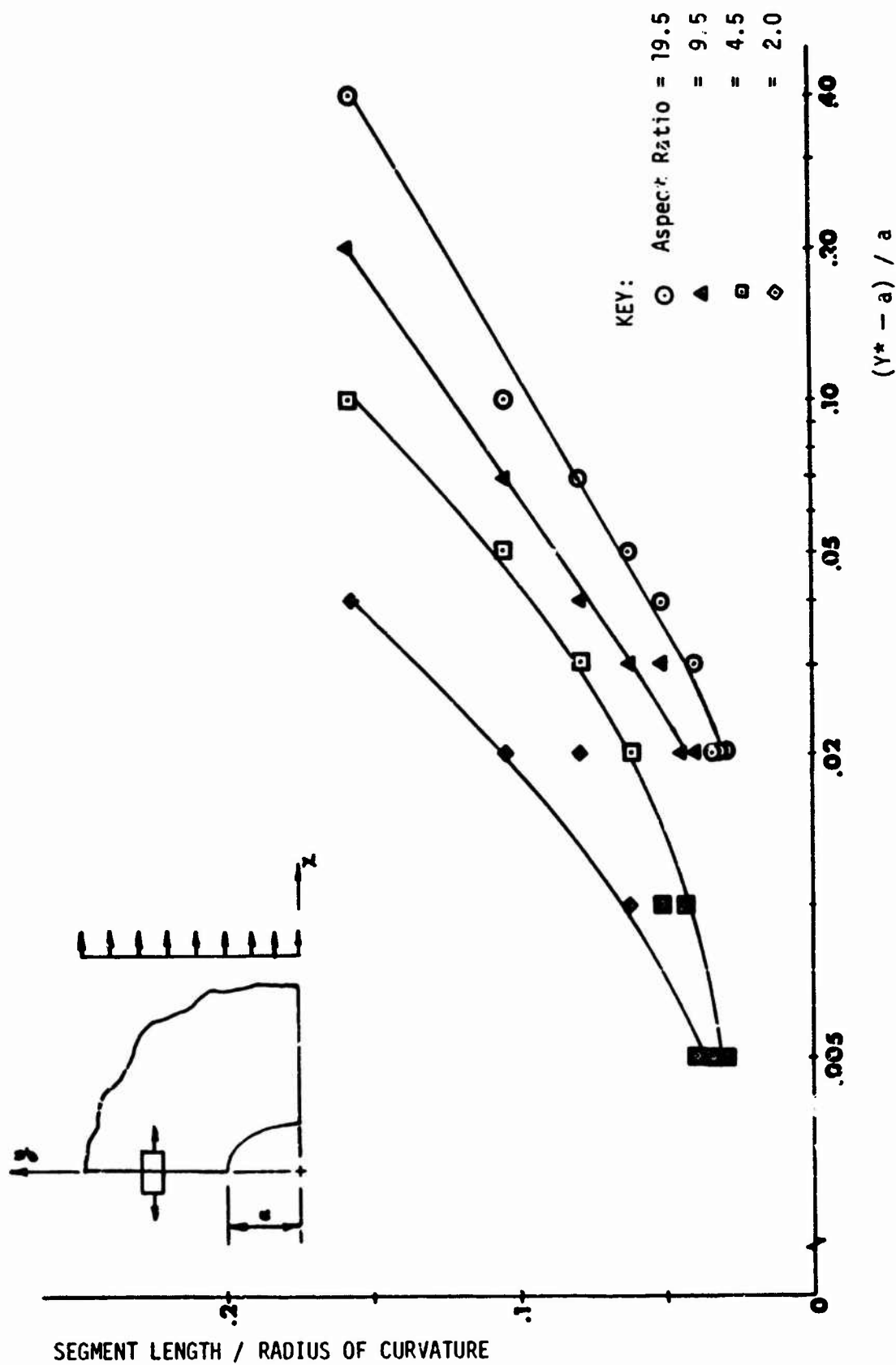
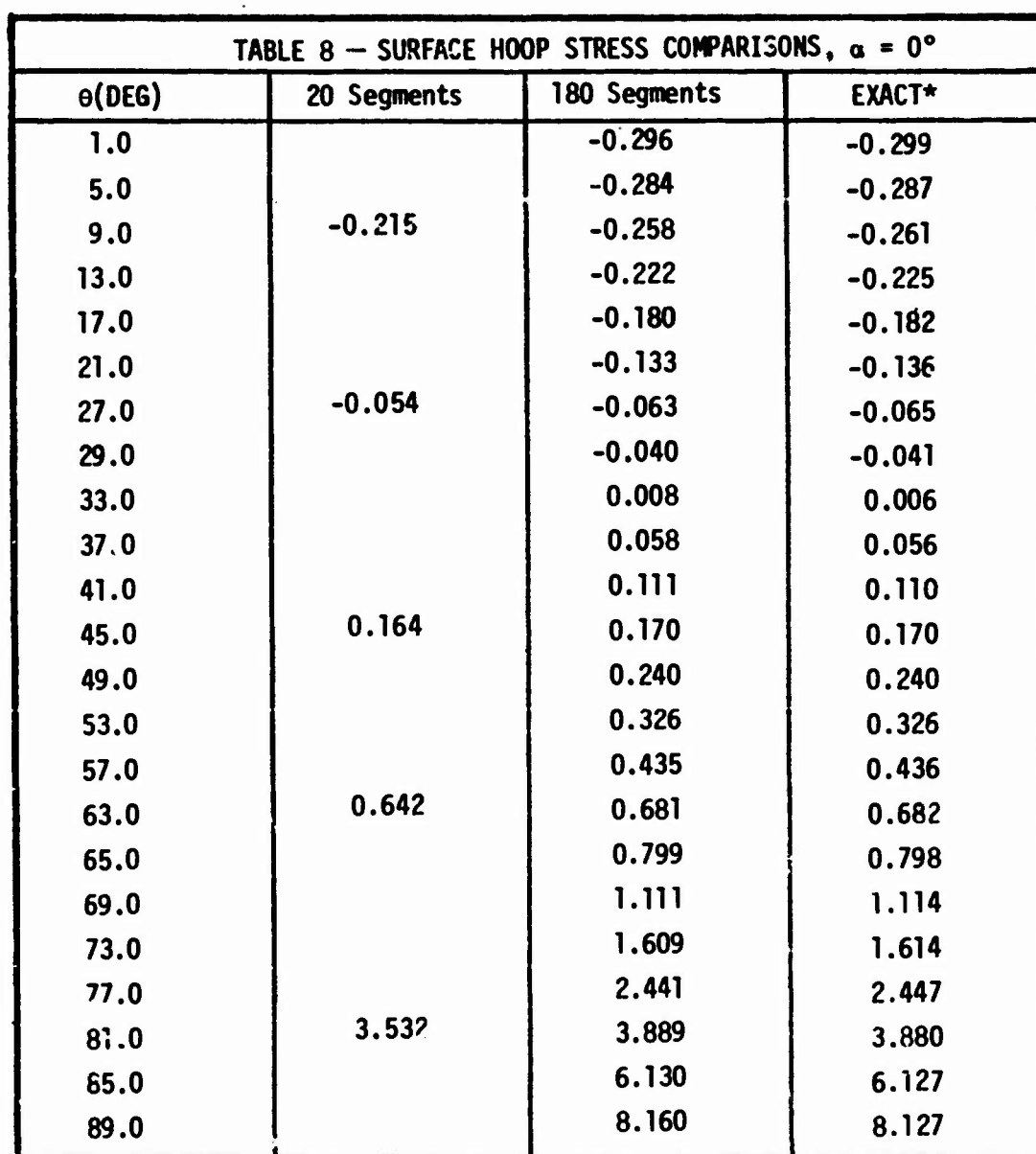


Figure 4: Accuracy of DIPOME — Elliptical Cutout



\*Due to Lekhnitskii

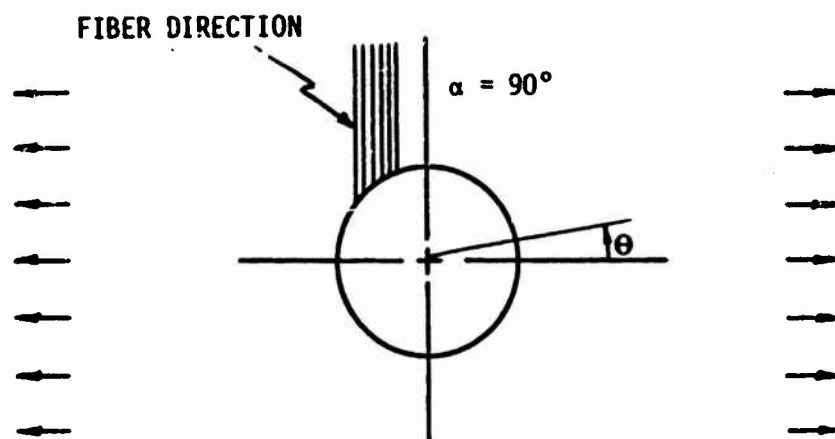


TABLE 9 — SURFACE HOOP STRESS COMPARISONS,  $\alpha = 90^\circ$

$\theta(\text{DEG})$	20 Segments	180 Segments	EXACT*
1.0		-3.19	-3.28
5.0		-2.22	-2.29
9.0	-0.367	-1.12	-1.165
13.0		-0.419	-0.442
17.0		+0.012	-0.008
21.0		0.281	+0.267
27.0	+0.625	0.546	0.538
29.0		0.616	0.608
33.0		0.741	0.736
37.0		0.859	0.856
41.0		0.979	0.976
45.0	+1.126	1.105	1.103
49.0		1.243	1.242
53.0		1.398	1.398
57.0		1.573	1.573
63.0	+1.892	1.877	1.878
65.0		1.990	1.991
69.0		2.229	2.230
73.0		2.478	2.481
77.0		2.725	2.725
81.0	+2.883	2.937	2.940
85.0		3.093	3.096
89.0		3.165	3.169

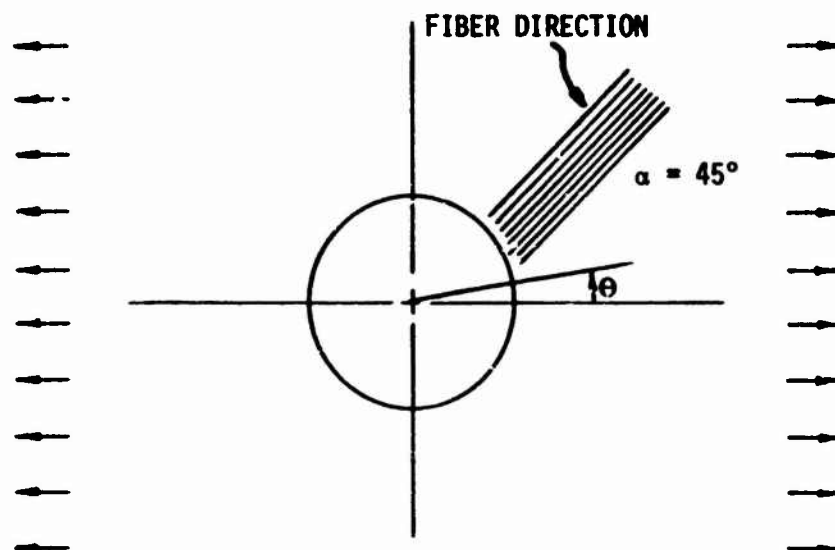


TABLE 10 — SURFACE HOOP STRESS COMPARISONS,  $\alpha = 45^\circ$

THETA	EXACT	90 Segments	THETA	EXACT	90 Segments
0	-0.812	-0.808	96	2.168	2.169
4	-0.706	-0.703	104	2.396	2.400
8	-0.596	-0.593	112	2.840	2.848
16	-0.337	-0.335	120	3.559	3.681
24	+0.015	+0.016	128	4.701	4.791
32	.499	.500	136	1.703	1.823
40	1.082	1.084	144	-1.799	-1.773
48	1.620	1.623	152	-1.804	-1.787
56	1.955	1.957	160	-1.444	-1.434
68	2.081	2.082	168	-1.151	-1.144
72	2.073	2.073	176	-0.918	-0.913
80	2.051	2.052	184	-0.706	-0.703
88	2.069	2.070			

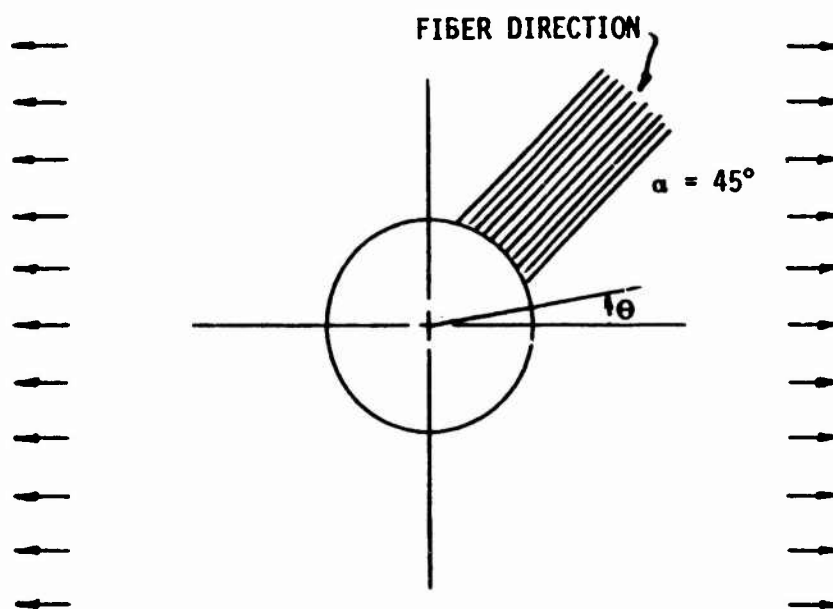


TABLE 11 - SURFACE HOOP STRESS COMPARISONS		
THETA	EXACT	20 Segments
9	-0.567	-0.565
27	0.180	0.186
45	1.436	1.477
63	2.067	2.107
81	2.050	2.065
99	2.235	2.246
117	3.296	3.473
135	2.453	3.189
153	-1.759	-1.747
171	-1.058	-1.043
189	-0.567	-0.565

## 5.4 ADVANCED TOPICS IN ANISOTROPIC INTEGRAL EQUATION SOLUTION METHODS

### 5.4.1 Introduction

The "integral equation method" referred to in this Section is basically a technique for obtaining accurate approximate solutions for a wide variety of physical problems governed by linear partial differential equations. As is clear from a number of papers e.g. [1,2,3,4,5] the method has reached a considerable stage of development and is emerging as an important tool comparable to and potentially, we think, better than finite element and finite difference techniques for certain problems. This appears to be particularly true for a variety of problems involving material composites.

A glance at the work cited above reveals that the method depends crucially on the existence and explicit definition of a fundamental singular solution to the appropriate governing partial differential equations. Therefore, in an attempt to open up the field of linear *three-dimensional anisotropic* elasticity to attack, via the integral equation method, considerable effort was directed toward investigating what is known of the necessary singular solution. As noted earlier, this solution is the field due to a concentrated force in an infinite anisotropic media. Two major works [6] and [7] were found on this topic, and examined with respect to the stated objective. Details primarily concerned with making representations of the singular solution available for practical purposes are given later in this Section.

The problem of interlaminar shear was investigated with a view toward attacking this problem (as defined by R. B. Pipes [8]) via the integral equation method. Under the appropriate assumptions, the

relevant surface integrals reduce to path integrals around each of the layers. While the problem is not completely two-dimensional in nature, significant advantages still seem to be present with the integral equation method for both isotropic and anisotropic layers to warrant further investigation with test problems. Details of the formulation for isotropic layer assumptions and a discussion of the possibilities for anisotropic layers are included in this Section.

#### 5.4.2 *Fundamental Three-Dimensional Anisotropic Singularity*

##### 5.4.2.1 *Via John [7]*

The work by John [7] which is in essence "a somewhat heterogeneous collection of results on partial differential equations" contains, in Ch. III, a method for constructing the so-called *fundamental singular solution* for an elliptic system of linear partial differential equations with analytic coefficients. Since our main concern here is with homogeneous anisotropic elasticity theory, we will specialize John's development at the outset and explicitly deal with the system of equations

$$C_{ijkl}u_{k,lj} = 0 \quad (1)$$

Equations (1) are the equations of equilibrium in the absence of body forces for a linear elastic solid obeying the constitutive relation

$$\tau_{ij} = C_{ijkl}e_{kl} \quad (2)$$

in which  $C_{ijkl}$  are constants, and  $\tau_{ij}$ ,  $e_{ij}$ ,  $u_i$  the stress, strain, and displacement components assuming the linearized theory. As usual we take

$$C_{ijkl} = C_{klij} \quad (3)$$

together with necessary symmetries in the first and second pair of indices such that  $C_{ijkl}$  implies at most twenty one independent constants.



We recognize that Eqs. (1) imply the existence of a set of differential operators  $L_{ij}$  such that Eqs. (1) may be written, for convenience in the symbolic form

$$L_{ik}[u_k] = 0 \quad (4)$$

where, specifically,

$$L_{ik} = C_{ijkl} \frac{\partial^2}{\partial x_l \partial x_j} \quad (5)$$

A fundamental system of singular solutions  $U_{jk}$  of Eqs. (4) according to John [7] has property that the symbolic equations

$$L_{ik}[U_{jk}(\underline{x}, \underline{y})] = 0 \quad \text{for } \underline{x} \neq \underline{y} \quad (6)$$

are satisfied where  $\underline{x}$  and  $\underline{y}$  are two arbitrary points in space.

Further, the functions  $U_{jk}$  have the additional property that

$$\int_{\partial R + \Gamma} [t_j(\underline{y}) U_{jk}(\underline{x}, \underline{y}) - u_j(\underline{y}) T_{jk}(\underline{x}, \underline{y})] da(\underline{y}) = 0 \quad (7)$$

where  $\partial R$  is the boundary of a regular region of space  $R$ , and  $\Gamma$  is the surface of a small sphere of radius  $\epsilon$  surrounding the point  $\underline{x}$ ,  $n_k$  are the components of the "outer" normal at  $\underline{y}$  ( $\underline{y}$  on  $\partial R + \Gamma$ ) to the region "enclosed by"  $\partial R + \Gamma$ , and  $T_{jk}$  represents a set of functions derivable from  $U_{jk}$ . The function  $u_j$  is an arbitrary solution to Eq. (4) and  $t_j$ , derivable from  $u_j$ , represents the surface traction on the anisotropic body which is assumed to occupy the region  $R$ . If we now take the limit in Eq. (7) as  $\epsilon$  goes to zero, i.e., shrink  $\Gamma$  indefinitely about  $\underline{x}$ , the orders of the singularities in  $U_{jk}$  and  $T_{jk}$  are such that Eq. (7) reduces to

$$u_k(\underline{x}) = \int_{\partial R} [u_j(\underline{y}) T_{jk}(\underline{x}, \underline{y}) - t_j(\underline{y}) U_{jk}(\underline{x}, \underline{y})] da(\underline{y}) \quad (8)$$

A glance at the cited works [1,3,4,5] reveals that the above properties (6), (7), and (8) of the functions  $U_{jk}$  are precisely those needed to formulate the integral equation method for three-dimensional anisotropic elastic boundary value problems. Physically,  $U_{jk}$  represents a set of displacement or influence functions; i.e.,  $U_{jk}(\underline{x}, \underline{y})$  is the displacement in the  $j$  coordinate direction at  $\underline{y}$  due to a concentrated force in the  $k$  coordinate direction at  $\underline{x}$ . Further,  $T_{jk}$  represents traction components at  $\underline{y}$  across an arbitrary surface with orientation  $\underline{n}$ . These are obtained from  $U_{jk}$  according to the familiar relation

$$T_{jk} = \frac{1}{2} C_{jplm} [U_{lk,m} + U_{mk,l}] n_p \quad (9)$$

just as the arbitrary traction  $t_j$  is related to  $u_j$  according to

$$t_j = \frac{1}{2} C_{jplm} [u_{l,m} + u_{m,l}] n_p \quad (10)$$

Thus since the relation (8) is the desired relation to accomplish the anisotropic formulation everything depends on the availability of an explicit relation for  $U_{jk}$ . To construct  $U_{jk}$  for Eqs. (4), with the properties (6) through (8) John [7], pg. 76, gives the formula

$$U_{jk}(\underline{x}, \underline{y}) = \frac{-1}{16\pi^2} \Delta_{\underline{y}} \int_{\Omega_{\underline{\xi}}} P^{kj}(\underline{\xi}) [(\underline{x}-\underline{y}) \cdot \underline{\xi}] \text{sgn}[(\underline{x}-\underline{y}) \cdot \underline{\xi}] d\Omega_{\underline{\xi}} \quad (11)$$

In Eq. (11),  $\Delta_{\underline{y}}$  is the Laplacian with respect to the coordinates at  $\underline{y}$  of the integral over  $\Omega_{\underline{\xi}}$  which is a sphere of unit radius with origin at  $\underline{\xi} = 0$ .  $P^{kj}(\underline{\xi})$  is the inverse of the matrix  $Q_{ik}(\underline{\xi})$  which in turn is the characteristic form of the operator  $L_{ik}$ . This characteristic form is explicitly

$$Q_{ik}(\underline{\xi}) = C_{ijkl} \xi_l \xi_j \quad (12)$$

in which  $\xi_i$  are components of the vector  $\underline{\xi}$ .

Space does not permit nor would it be appropriate to discuss here the rather detailed, abstract, and frequently obscure arguments leading up to formula (11). Moreover, formula (11) as written above is an abridgement of the relations which appear in John [7] appropriate to anisotropic elasticity with the additional assumption of material homogeneity ( $C_{ijkl}$  constants). The actual treatment in John [7] deals with operators of more general order than two and in spaces of  $n$  dimension as well as allowing for the possibility of non-constant (but analytic) coefficients. This last feature could be of interest for problems involving inhomogeneous media. However, the remainder of the present discussion will be confined to  $U_{jk}$  as given by formula (11). Indeed, as will be explained, algebraic expressions for  $U_{jk}$  from Eq. (11) even under the present assumptions of *full* (21 constant) *anisotropy* will be difficult to obtain.

To best appreciate the last remark consider now formula (11) in more detail. Let  $\underline{x} - \underline{y} \equiv \underline{R}$  such that

$$\underline{R} \cdot \underline{\xi} \equiv \text{sgn } \underline{R} \cdot \underline{\xi} \equiv R |\cos \phi| \quad (13)$$

where  $R$  is the magnitude, i.e.,  $R \equiv |\underline{R}|$ , of  $\underline{R}$ ,  $\xi = |\underline{\xi}| = 1$ , and  $\phi$  is the angle between  $\underline{R}$  and  $\underline{\xi}$ . Thus since  $R$  does not vary with  $\underline{\xi}$  Eq. (11) may be written

$$U_{jk}(\underline{x}, \underline{y}) = \frac{-1}{16\pi^2} \Delta_{\underline{y}} \left\{ R \int_{\Omega_{\underline{\xi}}} p^{kj}(\underline{\xi}) |\cos \phi| d\Omega_{\underline{\xi}} \right\} \quad (14)$$

Further, let

$$A_{jk} \equiv \int_{\Omega_{\underline{\xi}}} p^{kj}(\underline{\xi}) |\cos \phi| d\Omega_{\underline{\xi}} \quad (15)$$

such that

$$U_{jk}(\underline{x}, \underline{y}) = -\frac{1}{16\pi^2} \Delta_y \{RA_{jk}\} \quad (16)$$

Clearly  $A_{jk}$  is a tensor whose components depend only on  $C_{ijkl}$  and, of course, the choice of cartesian basis inasmuch as the components of  $U_{jk}$  itself depend on such a basis. Thus the ability to obtain explicit algebraic expressions for  $U_{jk}$  is dependent solely upon the ability to perform the integrations (15) for  $A_{jk}$ .

As mentioned earlier,  $p^{kj}(\underline{\xi})$  is the inverse of the quadratic form  $Q_{ik}(\underline{\xi})$  (Eq. 12). Explicitly,

$$p^{kj}(\underline{\xi}) = \frac{\frac{1}{2} \epsilon_{klm} \epsilon_{jpn} Q_{lp}(\underline{\xi}) Q_{mq}(\underline{\xi})}{\text{Det } Q} \quad (17)$$

in which  $\epsilon_{ijk}$  is the alternating symbol and  $\text{Det } Q$  is the determinant of the matrix  $Q_{ik}$ . Now since  $\text{Det } Q$  is of sixth degree in  $\epsilon_i$  and the numerator in (17) is of fourth degree, the ability to evaluate the elements  $A_{jk}$  analytically in closed form is largely dependent on the ability to factor the expressions implied by (17). Guided by the related investigations of Kroner [10] and Lie and Koehler [11] this is expected to be possible under the assumptions of special anisotropy, e.g., cubic or hexagonal symmetry. However, recognition of the tensor character of  $A_{jk}$  allows the following plan to be adopted in order to obtain explicit practical expressions for  $U_{jk}$  under more general conditions of anisotropy. Choose a convenient orthonormal basis and evaluate, *numerically* if need be, the integrals in (15) for a given set of  $C_{ijkl}$ . Having thus obtained a set of values for  $A_{jk}$  for that basis,  $A_{jk}$  for any other basis is obtainable by simple cartesian tensor transformation. Recognizing further that the direction

cosines of  $\underline{R}$  referred to a given basis are of the form  $[x_j(\underline{x}) - x_j(\underline{y})]/R$ , allows the gradient and Laplacian with respect to  $\underline{y}$  to be evaluated as required in Eq. (16).

As an example of the above consider the special case of complete isotropy for which

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk}) \quad (18)$$

where  $\lambda$  and  $\mu$  are the Lamé elastic constants. Here it is easily shown through Eq. (17) that  $p^{kj}(\underline{\xi})$  has the form

$$p^{jk}(\underline{\xi}) = \alpha (\delta_{ij} + \beta \xi_i \xi_j)$$

where  $\alpha$  and  $\beta$  are constants obtainable from  $\lambda$  and  $\mu$  alone. Thus the expressions for  $A_{jk}$  via Eq. (15) become

$$A_{jk} = \alpha \delta_{jk} \int_{\Omega_{\underline{\xi}}} |\cos \phi| d\Omega_{\underline{\xi}} + \alpha \beta \int_{\Omega_{\underline{\xi}}} \xi_j \xi_k |\cos \phi| d\Omega_{\underline{\xi}} \quad (19)$$

A little reflection on the integrals in Eq. (19) reveals that the first integral is twice the first moment of a unit hemispherical shell about the basal plane perpendicular to  $\underline{R}$ . Similarly, the second integral represents the inertia components of a spherical shell referred to a given basis where the "mass density" ( $|\cos \phi|$ ) of the shell varies linearly with respect to height above the same basal plane. Clearly, the calculations here would most conveniently be made taking one coordinate direction in the direction of  $\underline{R}$  and the other two in the mentioned basal plane. Subsequently, the desired  $A_{jk}$  for a more general orientation of basis with respect to  $\underline{R}$  could be easily obtained by cartesian tensor transformation. We note finally in passing that the result of the above calculations for material isotropy results in

$$U_{jk}(\underline{x}, \underline{y}) = \frac{1}{16\pi\mu(1-\nu)R} [(3-4\nu)\delta_{jk} + \cos \psi_j \cos \psi_k] \quad (20)$$

in which  $\nu = \lambda/2 (\lambda + \mu)$  and  $\psi_j$  is the angle between the vector  $\underline{R}$  and the  $x_j$  axis. Expression (20) is the fundamental isotropic singular solution (see e.g. Cruse [3]).

The key feature of the above proposed method is the ability to perform, if need be, part of the calculation numerically and still obtain all dependence of  $U_{jk}$  on  $\underline{x}, \underline{y}$  and basis orientation with respect to  $\underline{x}-\underline{y}$  analytically. This is important since gradients of  $U_{jk}$  at  $\underline{y}$  are required for the integral equation method and such gradients may therefore be taken analytically. Thus, in light of the goal of this portion of the research, i.e., obtaining an explicit, usable, algebraic form for  $U_{jk}$  for complete anisotropy, it appears, despite numerical evaluation of certain integrals in general, that the job can be done via the outlined method.

#### 5.4.2.2 Via Fredholm [6]

The fundamental paper by Fredholm [6] displays an alternative method for constructing, in principle, the fundamental solution  $U_{jk}$  discussed above. Like John's [7], Fredholm's work leads to a formal implicit representation for the solution. However, unlike with John's procedure it is not clear to the writer that one would be able to effect as useful a reduction of the method except for special anisotropy, by any means numerical or otherwise

Fredholm motivates his work by attempting to extend the idea of the particular solution  $1/r$  of Laplace's equation  $\Delta u = 0$  to the equations of anisotropic elasticity (1). He first eliminates two components of  $u_k$  in Eqs. (4) and shows that the remaining component (and hence each component  $u_k$ ) must satisfy a sixth-order differential equation of the form

$$f(u_k) = 0 \quad (21)$$

where  $f$  is a sixth-order linear homogeneous differential operator which is explicitly the determinant of  $L_{ik}$  (Eq. (5)). He then chooses as his fundamental solution

$$u_i = \int_C \frac{\psi_i(\xi, \eta) d\xi}{f_2(\xi, \eta) (\xi x_1 + \eta x_2 + x_3)} \quad (22)$$

where  $\psi_i$  are polynomials in  $\xi$  and  $\eta$  of the fifth order or lower and

$$f_2(\xi, \eta) \equiv \frac{\partial}{\partial \eta} f(\xi, \eta, 1), \quad (23)$$

with  $f(\xi, \eta, 1)$  being the definite algebraic form obtained by replacing the operations  $\partial/\partial x_i$ , ( $i = 1, 2, 3$ ) by  $\xi, \eta$ , and  $1$ , respectively. The

integration is around a closed contour  $c$  in  $\xi$  space containing only those singular points which are roots of  $f(\xi, \eta_0) = 0$  where  $\eta_0$  is given by

$$\xi x_1 + \eta_0 x_2 + x_3 = 0 \quad (24)$$

The polynomials  $\psi_i$  above are complicated algebraic expressions (see [6] pg. 14) obtained from  $L_{ijk}$ . Fredholm then goes on to show that each component of the required tensor field  $U_{jk}$  is of the form (22) and rigorously establishes all of the properties of the solution.

It seems clear from the work of Kroner [10] and Lie and Koehler [11] that any attempt to reduce Fredholm's method to something useful for other than hexagonal or cubic crystal symmetry assumptions would be most difficult indeed. Detailed information on the particulars of this can best be obtained by careful study of the references [10,11] plus Fredholm's original paper [6]. It should be clear; however, that if the previous discussion and reduction of John's [7] approach is valid as outlined, it must be possible to accomplish the same task via Fredholm also since the desired  $U_{jk}$  is unique. Nevertheless, the transformation of contour integral in space to one over the unit sphere, of functions which are necessarily related but not explicitly so, is bound to be an extremely difficult task. Further, for *practical* purposes and in light of the previous section the effort seems hardly worthwhile in the near future.

It is my judgement that to formulate the integral equation method for general anisotropic elasticity, the method of John as previously outlined is by far the most promising at this point. Indeed, the outlined reduction with the ability to obtain the necessary functional variational analytically is better than was hoped for at the start of the



investigation. If a similar advantage plus others are present also in Fredholm's technique they are lost to me, although, to be fair, much more time was spent with [7] than [6] because of the positive indications of [7].

#### 5.4.3 Investigation of the Interlaminar Shear Problem

Consider a laminated plate as shown in Fig. 1 loaded on its "x" faces in such a way (cf. Pipes [8]) that it may be assumed that the stress and strain fields are functions of y and z alone. Further it is assumed that displacement components are of the form

$$\begin{aligned} u_1 &= cx + U_1(y, z) \\ u_2 &= U_2(y, z) \\ u_3 &= U_3(y, z) \end{aligned} \quad (25)$$

where  $U_i$ , are arbitrary functions and c is a constant. Finally, under the assumption that each lamina is homogeneous and isotropic it is now desired to examine the possible simplifications which may arise with the integral equation method by the process of "integrating out" dependence on x.

Specifically, consider the boundary formula of Cruse ([3] Eq. (14)) written for a typical lamina

$$\frac{1}{2} u_j(P) + \int_S u_i(Q) T_{ji}(P, Q) dS(Q) = \int_S t_i(Q) U_{ji}(P, Q) dS(Q) \quad (26)$$

where explicitly S is the union of surfaces  $S_x$ ,  $S_y$ ,  $S_z$  of the lamina perpendicular to the x, y, z directions, respectively, see Fig. 2. Clearly, each integral over  $S_x$  is an integral of functions of ( $\pm \ell$ , y, z) such that there is no explicit x dependence to be "removed" in those integrals. Further, since there is assumed to be no traction on the  $S_y$  surfaces we have

$$\int_{S_y} t_i(Q) U_{ji}(P, Q) dS(Q) \equiv 0 \quad (27)$$

Thus, it remains to consider the integrals

$$\int_{S_x + S_z} u_i(Q) T_{ji}(P, Q) dS(Q), \int_{S_z} t_i(Q) U_{ji}(P, Q) dS(Q) \quad (28)$$

insofar as integrating away the  $x$  dependence. More explicitly, integrals (28) may be written

$$\int_{-w}^w t_i(y, \pm t) \int_{-\ell}^{\ell} U_{ji}(x, y, \pm t; \xi, \eta, \zeta) dx dy \quad (29)$$

$$\int_{-w}^w u_i(y, \pm t) \int_{-\ell}^{\ell} T_{ji}(x, y, \pm t; \xi, \eta, \zeta) dx dy \quad (i \neq 1) \quad (30)$$

$$\int_{-w}^w u_1(y, \pm t) \int_{-\ell}^{\ell} T_{ji}(x, y, \pm t; \xi, \eta, \zeta) dx dy + \oint_{-w}^w \int_{-\ell}^{\ell} x T_{ji}(x, y, \pm t; \xi, \eta, \zeta) dx dy \quad (31)$$

$$\int_{-t}^t u_i(\pm w, z) \int_{-\ell}^{\ell} T_{ji}(x, \pm w, z; \xi, \eta, \zeta) dx dz \quad (i \neq 1) \quad (32)$$

$$\int_{-t}^t u_1(\pm w, z) \int_{-\ell}^{\ell} T_{ji}(x, \pm w, z; \xi, \eta, \zeta) dx dz + \oint_{-t}^t \oint_{-\ell}^{\ell} x T_{ji}(x, \pm w, z; \xi, \eta, \zeta) dx dz \quad (33)$$

in which  $x, y, z$  are the coordinates of the point  $Q$  and  $\xi, \eta, \zeta$  are the coordinates of the point  $P$ . Our task therefore is to examine the expressions for each component of the kernel functions  $U_{ij}$  and  $T_{ij}$  as given by Eqs. (5) and (7) in Cruse [3], and then perform the definite integrals

with respect to  $x$  alone from  $-\ell$  to  $\ell$  as indicated in expressions (29) through (33) above. Note that performing the first integration with respect to  $x$  in expressions (31) and (33) will suffice since the second such integration is obtainable directly from the first by parts.

Careful consideration of the mentioned Eqs. (5) and (7) in [3] for the components of  $U_{ij}$  and  $T_{ij}$  and designating all parts of the integrands which are independent of  $x$  with the common symbol  $B$  leads, after some "bookkeeping", to the need to evaluate only integrals of the following type

$$\int_{-\ell}^{\ell} \frac{(x - \xi)^n dx}{[(x - \xi)^2 + B^2]^{m/2}} \quad (34)$$

where  $n$  takes on integer values from zero through 3 and  $m$  takes on integer values 1, 3 and 5. Such integrals for values of  $m$  and  $n$  indicated are standard entries in any short table of integrals and result in polynomial and/or logarithmic forms in the variable  $(x - \xi)$ .

Maintaining care with the mentioned bookkeeping problem, and recognizing that each of the  $-\ell$  to  $\ell$  integrations in expressions (29) through (33) result in new tensor functions  $\hat{U}_{ji}$ ,  $\hat{T}_{ji}$  independent of  $x$ , we may write the boundary formula (26) in the form

$$\begin{aligned} & \frac{1}{2} u_j(\xi, n, \zeta) + \int_{-t}^t U_i(\pm w, z) \hat{T}_{ji}(\pm \ell, \pm w, z, \xi, n, \zeta) dz \\ & + \int_{-w}^w U_i(y, \pm t) \hat{T}_{ji}(\pm \ell, y, \pm t; \xi, n, \zeta) dy \\ & - \int_{-w}^w t_i(y, \pm t) \hat{U}_{ji}(\pm \ell, y, \pm t; \xi, n, \zeta) dy = \end{aligned} \quad (35)$$

$$\int_{S_x} [t_i(y,z) U_{ji}(\pm \ell, y, z; \xi, \eta, \zeta) U_i(y,z) T_{ji}(\pm \ell, y, z, \xi, \eta, \zeta)] dy dz \quad (35)$$

$$+ c f_j(\pm \ell, \pm w, \pm t; \xi, \eta, \zeta)$$

where  $f_j(\xi, \eta, \zeta)$  is that function obtained by integrating *all*<sup>1</sup> terms  $T_{ji}$  with  $c$  as a constant multiplier.

The question now arises, to what extent may explicit dependence on the length  $\ell$  of the lamina be eliminated and still retain sufficient information to obtain what is required in a given problem. Examination of the terms in equation (35) reveals that as  $\ell$  goes to infinity,  $f_j$  is bounded, and all components of  $U_{ji}$  and  $T_{ji}$ , with the exception of  $U_{11}$ , are zero or finite. The  $U_{11}$  component, which alone contains logarithmic terms blows up with increasing  $\ell$ . However, since it may be argued that the component of traction  $t_1$  on the surfaces  $S_z$  must be zero for isotropic media under the present assumptions, no difficulty is, in fact, encountered with that term. Finally, it is clear that the integrals over  $S_x$  on the right side of Eq. (35) vanish with increasing  $\ell$ , such that all "input" information on the faces  $S_x$  indefinitely far apart is contained in the limit of the term  $c f_j$ .

It is now evident that it suffices to consider the "mid-x" plane of a typical lamina and to allow point  $P$  to occupy positions only on the rectangular boundary of this plane (i.e., consider only  $\xi = 0$ ). Thus, Eq. (35) in reduced,  $x$ -independent, form may be written

<sup>1</sup>Note that one term is contributed to  $f_j$  from each pair of surfaces of the lamina.

$$\frac{1}{2} U_j(n, \zeta) + \int_{-t}^t U_i(\pm w, z) \hat{T}_{ji}'(\pm w, z, n, \zeta) dz - \int_{-w}^w t_i(y, \pm t) \hat{U}_{ji}'(y, \pm t; n, \zeta) dy \quad (36)$$

$$+ \int_{-w}^w U_i(y, \pm t) \hat{T}_{ji}'(y, \pm t, n, \zeta) dy = C f_j'(\pm w, \pm t, n, \zeta)$$

where the primes indicate limiting forms of the functions as  $\ell \rightarrow \infty$  and  $\xi = 0$ .

Application of Eq. (36) in the solution of the interlaminar shear problem is as follows. Specify the constant  $c$  and perform the necessary integrations to obtain the function  $f_j'(\pm w, \pm t, n, s)$  for each lamina mid-plane. Then, write Eq. (36) for each such plane using an appropriate approximation scheme as, perhaps, outlined by [1,4]. Recognize further that the two integrals in Eq. (36) from  $-w$  to  $w$  for a given lamina mid-plane are coupled with similar integrals for the remaining lamina. The boundary conditions between lamina are that  $U_i$  and  $t_i$  be continuous across the adjacent boundaries and that the top and bottom boundaries are free of traction  $t_i$ . Unknowns to be obtained therefore by numerical solution of the integral equations are discrete values of  $U_i(y, z)$  and  $t_i(y, z)$  at selected discrete points on the boundaries of the lamina mid-planes.

Note in the above that while the integrations in Eq. (36) are over the lamina mid-plane boundaries all indices have the range 1, 2, 3 such that as mentioned in the introduction the problem is not truly two-dimensional in nature. However, it appears that the method outlined above is most feasible with much promise for success in light of numerical

work already accomplished for both two and three dimensional problems (e.g. [2,3,9,12]). Most important, coupling the above ideas with those set forth in the previous section, it is possible to attack the difficult interlaminar shear problem under the assumption of fully anisotropic or specially anisotropic lamina.

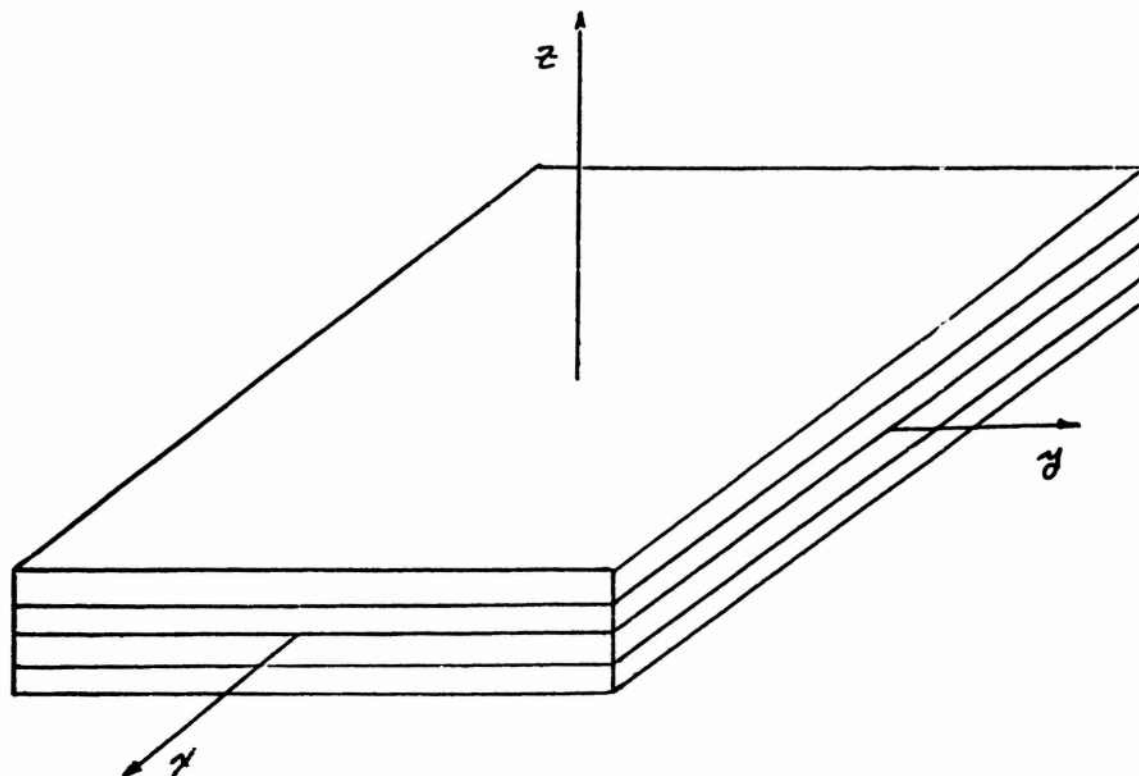


FIGURE 1: GEOMETRY FOR INTERLAMINAR SHEAR PROBLEM

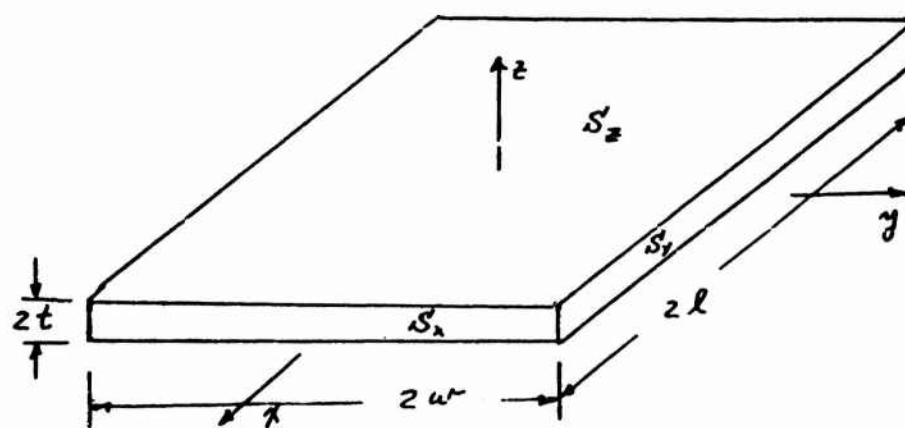


FIGURE 2: INDIVIDUAL LAMINA NOTATION

#### 5.4.4 References

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